UNIT 5

Algebra

Introduction

In Unit 2 you met the idea of using letters to represent numbers. In this unit you'll learn much more about this sort of mathematics, which is called **algebra**. You'll begin to see how useful it can be, and you'll begin to learn the techniques that will allow you to make the most of it.

You'll need to use the algebraic skills covered in this unit when you study some of the later units in the module, and they're essential for any further mathematics modules. So it's important that you become proficient in them. The only way to do that is to practise them! You'll have lots of opportunity for that in this unit.

The word 'algebra' is derived from the title of the treatise *Al-kitab-al mukhtasar fi hisab al-jabr* (Compendium on calculation by completion and reduction), written by the Central Asian mathematician Muḥammad ibn Mūsā al-Khwārizmī in around 825. This treatise deals with solving linear and quadratic equations, which you'll learn about in this module, starting with linear equations in this unit. The treatise doesn't use algebra in the modern sense, as no letters or other symbols are used to represent numbers. Modern algebra developed gradually over time, and it was not until the sixteenth and seventeenth centuries that it emerged in the forms that we recognise and use today.

Muḥammad ibn Mūsā al-Khwārizmī was a member of the House of Wisdom in Baghdad, a research centre established by the Caliph al-Ma'mūn. His name is the origin of the word 'algorithm'. As you saw in Unit 2, an algorithm is a procedure for solving a problem or doing a calculation.

1 Why learn algebra?

What's the point of learning algebra? Why is it useful? In this section you'll see some answers to these questions.

1.1 Proving mathematical facts

This first activity is about a number trick.

Activity 1 *Think of a number*

Try the following number trick.

Choose a fairly small number, so that the arithmetic is easy!

Think of a number.
Double it.
Add 7.
Double the result.
Add 6.
Divide by 4.
Take away the number you first thought of.
Find the corresponding letter of the alphabet.
Name an animal beginning with that letter.

For example, if your number is 3, then your letter is C, the third letter of the alphabet.

Now look at the solution on page 51.

Activity 2 *Think of another number*

Try the trick in Activity 1 again, choosing a different number to start with. Look at the solution on page 51 and then read the discussion below.

In Activities 1 and 2 you probably found that with both your starting numbers you obtained the number 5 in the third-last step (the last step involving a mathematical calculation) and so each time you obtained the letter E. If you didn't, then check your arithmetic! When asked to name an animal beginning with the letter E, nearly everyone thinks of 'elephant'.

You might like to try the trick on a friend, or a child who's old enough to do the arithmetic. Write the word 'elephant' on a piece of paper beforehand, ready to reveal at the end of the trick.

The idea behind the trick is that the number 5 is obtained in the last mathematical step, no matter what the starting number is. But how can you be sure that the trick works for *every possible* starting number? You can't test them all individually as there are infinitely many possibilities.

There's a way to check this – using algebra. In this unit you'll learn the algebraic techniques that are needed, and you'll see how to use them to check that the trick always works.

As you saw earlier in the module, a demonstration that a piece of mathematics *always* works is called a **proof**. Proofs of mathematical facts are needed in all sorts of contexts, and algebra is usually the way to construct them.

1.2 Finding and simplifying formulas

Suppose that a baker makes a particular type of loaf. Each loaf costs 69p to make, and is sold for £1.24. The baker sells all the loaves that he makes.

On a particular day, the baker makes 30 loaves. Let's calculate the profit that he makes from them. The total cost, in £, of making the loaves is

$$30 \times 0.69 = 20.70.$$

The total amount of money, in £, paid for the loaves by customers is

$$30 \times 1.24 = 37.20.$$

So the profit in £ is given by

$$37.20 - 20.70 = 16.50.$$

That is, the profit is £16.50.

On a different day, the baker might make a different number of loaves. It would be useful for him to have a formula to help him calculate the profit made from *any* number of loaves. To obtain the formula, we represent the number of loaves by a letter, say n, and work through the same calculation as above, but using n in place of 30.

The total cost, in £, of making the loaves is

$$n \times 0.69 = 0.69n.$$

Remember that if a letter and a number are multiplied together, then we omit the multiplication sign, and we write the number first. So, for example, we write

$$n \times 0.69$$

as

$$0.69n.$$

The total amount of money, in £, paid for the loaves by customers is

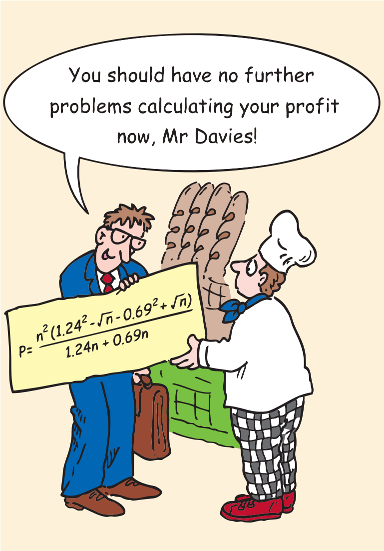
$$n \times 1.24 = 1.24n.$$

So if we represent the profit by £P, then we have the formula

$$P = 1.24n - 0.69n.$$

Activity 3 *Using a formula*

Use the formula above to calculate the profit for 48 loaves.



The formula makes it easy to calculate the profit, because you don't need to think through the details of the calculation. You just substitute in the number and do a numerical calculation. This is the advantage of using a formula.

In fact, the task of calculating the profit can be made even more straightforward. It's possible to find a *simpler* formula for P, by looking at the situation in a different way. The profit, in £, for each loaf of bread is

$$1.24 - 0.69 = 0.55.$$

So the profit, in £, for n loaves of bread is

$$n \times 0.55 = 0.55n.$$

So we have the alternative formula

$$P = 0.55n.$$

This formula, like the first one, can be used for any value of n.

Activity 4 *Using a better formula*

Use the new formula above to find the profit for 48 loaves of bread.

The alternative formula for the profit is better because it is simpler and using it involves less calculation.

In this case, a simpler formula was found by thinking about the situation in a different way. However, it is often easier to find whatever formula you can, and then use algebra to turn it into a simpler form. You'll begin to learn how to do this later in the unit.

Algebra can also help you to *find* formulas. The formula for the baker's profit was obtained directly from the situation that it describes, but it's often easier to obtain formulas by using other formulas that you know already. Algebra is needed for this process, and it's also needed to turn the new formula into a simpler form. You'll find out more about this in Unit 7.

1.3 Answering mathematical questions

Consider the following.

> A school has stated that 30% of the children who applied for places at the school were successful. It allocated 150 places. How many children applied?

To help you to think clearly about questions like this, it helps to represent the number that you want to find by a letter. Let's use N to represent the number of children who applied.

The next step is to write down what you know about N in mathematical notation. We know that 30% of N is 150. That is,

$$\frac{30}{100} \times N = 150,$$

which can be written more concisely as

$$\tfrac{3}{10}N = 150. \tag{1}$$

This is an example of an *equation*. To answer the question about the school places, you have to find the value of N that makes the equation correct when it's substituted in. This is called *solving* the equation.

You'll learn exactly what 'equation' means in the next section.

One way to solve the equation is as follows:

Three-tenths of N is 150,
so *one*-tenth of N is $150 \div 3 = 50$,
so N is $10 \times 50 = 500$.

You'll learn a different way to solve this equation in Section 5.

So the number of children who applied was 500.

You can confirm that this is the right answer by checking that equation (1) is correct when $N = 500$ is substituted in.

Activity 5 *Using an equation*

Two-fifths of the toddlers in a village attend the local playscheme. Twenty-four toddlers attend the playscheme. Let the total number of toddlers in the village be T.

(a) Write down an equation (similar to equation (1)) involving T.

(b) Find the value of T.

You could probably have answered the questions about the school places and the toddlers without using equations. But now read the following.

> Catherine wants to contribute to a charitable cause, using her credit card and a donations website. The donations company states that from each donation, first it will deduct a 2% charge for credit card use, then it will deduct a charge of £3 for use of its website, and then the remaining money will be increased by 22% due to tax payback. How much money (to the nearest penny) must Catherine pay if she wants the cause to receive £40?

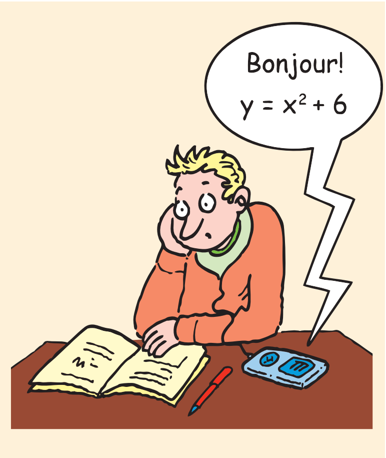
This question is a bit harder! But it can be answered by writing the information in the question as an equation and using algebra, as you'll see at the end of the unit.

The advantage of writing the information in a question as an equation is that it reduces the problem of answering the question to the problem of solving an equation. The equation may be more complicated than the two that you've seen in this section, but there are standard algebraic techniques for solving many equations, even complicated ones. You'll learn some of these techniques in this unit.

So now you've seen just the beginnings of what algebra can do. Algebra is used in many different fields, including science, computer programming, medicine and finance. For example, it's used to create formulas so that computer programs can carry out many different tasks, from calculating utility bills to producing images on screens. And it's used in mathematical

models, so that predictions, such as those about the economy and climate change, can be made by solving equations. Algebra allows us to describe, analyse and understand the world, to solve problems and to set up procedures. Our lives would be very different if algebra had not been invented!

You'll learn more about the power of algebra if you take further modules in mathematics.



1.4 How to learn algebra

Now that you've seen some reasons why algebra is useful, you should be ready to learn more about it.

Algebraic notation is 'the language of mathematics', and it takes time to learn, like any new language. So don't worry if you don't absorb some of the ideas immediately. Allow yourself time to get to grips with them, and keep practising the techniques. You learn a language by using it, not by reading about it! Remember that any difficulties will often be quickly sorted out if you call your tutor or post a question on the online forum.

The activities in this unit have been designed to teach you algebra in a step-by-step manner. They give you the opportunity to practise, and become familiar with, each new technique before you meet the next one – this is important, because most of the techniques build on techniques introduced previously. You should aim to do *all* the activities, and you should do them in the order in which they're presented.

Do the activities even if they look easy – many students find that there are small gaps or misunderstandings in their skills that they're unaware of until they attempt the activities or check their answers against the correct answers. This unit gives you the opportunity to identify and deal with such problems, and so prevent them causing difficulties later. You may even find that some activities throw new light on techniques with which you're familiar. Do *all* the parts of each activity – the parts are often different in subtle ways, and frequently the later parts are more challenging than the earlier ones.

Many activities are preceded by worked examples, which demonstrate the techniques needed. Before you attempt each activity, read through any relevant worked examples, or watch the associated tutorial clips if they're available, and try to make sure that you understand each step.

Tutorial clips are available for the more complex techniques. You will probably find that you learn more effectively by watching the tutorial clips rather than by just reading through the worked examples.

Set out your solutions in a similar way to those in the worked examples. Remember that any green text within the think bubble icons isn't part of the solutions, but any other words in the solutions *are* part of them, and similar explanations should be included in your own solutions.

Enjoy learning the new skills!

2 Expressions

In this section you'll learn some terminology used in algebra, and a useful technique – collecting like terms.

2.1 What is an expression?

In the module you've worked with various formulas, such as

$$P = 0.55n \quad \text{and} \quad T = \frac{D}{5} + \frac{H}{600}.$$

The first formula here is the 'baker's profit' formula from Subsection 1.2, and the second is Naismith's Rule from Unit 2.

These formulas involve the elements

$$P, \quad 0.55n, \quad T \quad \text{and} \quad \frac{D}{5} + \frac{H}{600},$$

which are all examples of *algebraic expressions*. An **algebraic expression**, or just **expression** for short, is a collection of letters, numbers and/or mathematical symbols (such as $+$, $-$, $\times$, $\div$, brackets, and so on), arranged in such a way that if numbers are substituted for the letters, then you can work out the value of the expression.

So, for example, $5n + 20$ is an expression, but $5n + \div 20$ is not an expression, because '$+\div$' doesn't make sense.

To make expressions easier to work with, we write them concisely in the ways you saw earlier in the module. In particular, we usually omit multiplication signs; things that are multiplied are just written next to each other instead. For example, $0.55 \times n$ is written as $0.55n$. But it's sometimes helpful to include some multiplication signs in an expression – and a multiplication sign between two numbers can't be omitted.

You saw how to write formulas concisely in Unit 2, Subsection 3.2.

When you're working with expressions, the following is the key thing to remember.

> Letters represent numbers, so the normal rules of arithmetic apply to them in exactly the same way as they apply to numbers.

In particular, the BIDMAS rules apply to the letters in expressions.

The BIDMAS rules were covered in Unit 1, Subsection 2.1 and are summarised in the Handbook.

When you substitute numbers for the letters in an expression and work out its value, you're **evaluating** the expression.

Example 1 *Evaluating an expression*

Evaluate the expression

$$4x^2 - 5y$$

when $x = 2$ and $y = -3$.

Solution

If $x = 2$ and $y = -3$, then

$$\begin{aligned} 4x^2 - 5y &= 4 \times 2^2 - 5 \times (-3) \\ &= 4 \times 4 - (-15) \\ &= 16 + 15 \\ &= 31. \end{aligned}$$

Activity 6 *Evaluating expressions*

Evaluate the following expressions when $a = -2$ and $b = 5$.

(a) $\frac{5}{2} + a$ (b) $-a + ab$ (c) ab^2 (d) $b + 3(b - a)$

Remember to apply the BIDMAS rules. Be particularly careful in part (d).

Expressions don't have to contain letters – for example, $(2+3)\times 4$ is an expression.

Every expression can be written in many different ways. For example, multiplication signs can be included or omitted. As another example, the expression $x+x$ can also be written as $2x$. That's because adding a number to itself is the same as multiplying it by 2.

For instance, $3+3$ means the same as 2×3.

If two expressions are really just the same, but written differently, then we say that they're different *forms* of the same expression, or that they're **equivalent** to each other. We indicate this by writing an equals sign between them. For example, because $x+x$ is equivalent to $2x$, we write

$$x+x=2x.$$

If two expressions are equivalent, then, whatever values you choose for their letters, the two expressions have the same value as each other.

Activity 7 *Checking whether expressions are equivalent*

Which of the following statements are correct?

(a) $u+u+u=3u$ (b) $a^2\times a=a^3$ (c) $2a\div 2=a$

(d) $p^2\times p^3=p^6$ (e) $z+2z=3z$ (f) $6c\div 2=3c$

(g) $a-b-2c=a+(-b)+(-2c)$ (h) $3n\div n=n$

You saw another example of equivalent expressions in Subsection 1.2, when two different formulas, $P=1.24n-0.69n$ and $P=0.55n$, were found for the baker's profit. Since the formulas must give the same values,

$$1.24n-0.69n=0.55n.$$

When we write an expression in a different way, we say that we're **rearranging**, *manipulating* or *rewriting* the expression. Often the aim of doing this is to make the expression simpler, as with the formula for the baker's profit. In this case we say that we're **simplifying** the expression.

The first known use of an equals sign was by the Welsh mathematician Robert Recorde, in his algebra textbook *The whetstone of witte*, published in 1557. He justified the use of two parallel line segments to indicate equality as follows: *bicause noe 2 thynges can be moare equalle.*

We use equals signs when we're working with expressions, but expressions don't *contain* equals signs. For example, the statements

$$x+x=2x \quad \text{and} \quad 1.24n-0.69n=0.55n$$

aren't expressions – they're equations. An **equation** is made up of *two* expressions, with an equals sign between them.

There's a difference between the two equations above and the ones that arose from the questions about school places and toddlers in Subsection 1.3. In that subsection, the number of children who applied to a school was found using the equation

$$\tfrac{3}{10}N=150.$$

This equation is correct for *only one* value of N (it turned out to be 500). In contrast, the equations

$$x+x=2x \quad \text{and} \quad 1.24n-0.69n=0.55n$$

are correct for *every* value of x and n, respectively. Equations like these, which are true for all values of the variables, are called **identities**. The different types of equation don't usually cause confusion in practice, as you know from the context which type you're dealing with.

There are similar differences in the use of letters to represent numbers. When the equation $\frac{3}{10}N = 150$ was used in Subsection 1.3, the letter N represented a *particular* number – it was just that we didn't know what that number was. This type of letter is called an **unknown**. In contrast, in the equation $x + x = 2x$ above, the letter x represents *any* number. A letter that represents any number (or any number of a particular type, such as any integer) is called a **variable**, as you saw in Unit 2. Usually you don't need to think about whether a letter is an unknown or a variable. Both types represent numbers, so the same rules of arithmetic apply in each case.

The first person to systematically use letters to represent numbers was the French mathematician François Viète. His treatise *In artem analyticem isagoge* (Introduction to the analytic art) of 1591 gives methods for solving equations, including ones more complicated than those in this module. Viète represented unknowns by vowels and known numbers by consonants (he represented known numbers by letters to help him describe the methods). However, he used words for connectives such as plus, equals and so on, and also to indicate powers. For example, he wrote '=' as 'aequatur', a^2 as 'a quadratus' and a^3 as 'a cubus'. So his algebra was still far from symbolic.

Viète also wrote books on astronomy, geometry and trigonometry, but he was never employed as a professional mathematician. He was trained in law, and followed a legal career for a few years before leaving the profession to oversee the education of the daughter of a local aristocratic family. His later career was spent in high public office, apart from a period of five years when he was banished from the court in Paris for political and religious reasons. Throughout his life, the only time he could devote to mathematics was when he was free from official duties.

Figure 1 François Viète (1540–1603)

2.2 What is a term?

Some expressions are lists of things that are all added or subtracted. Here's an example:

$$-2xy + 3z - y^2.$$

The things that are added or subtracted in an expression of this sort are called the **terms**. The terms of the expression above are

$$-2xy, \quad +3z \quad \text{and} \quad -y^2.$$

The plus or minus sign at the start of each term is *part of the term*. While you're getting used to working with terms, it can be helpful to mark them like this:

$$\underline{-2xy}\ \underline{+\,3z}\ \underline{-\,y^2}\,.$$

You need to make sure that the sign *at the start* of each term is included along with the rest of the term.

A sign *after* a term is part of the next term.

If the first term of an expression has no sign, then the term is added to the other terms, so really it has a plus sign – it's just that we normally don't write a plus sign in front of the first term of an expression. For example, if you have the expression

$$4a + c - 7\sqrt{b} - 5,$$

then you could write a plus sign at the start and mark the terms like this:

$$\underline{+4a}\ \underline{+c}\ \underline{-7\sqrt{b}}\ \underline{-5}\,.$$

Its terms are

$$+4a, \quad +c, \quad -7\sqrt{b} \quad \text{and} \quad -5.$$

Activity 8 *Identifying terms of expressions*

For each expression below, copy the expression, mark the terms and write down a list of the terms.

(a) $x^3 - x^2 + x + 1$ (b) $2mn - 3r$ (c) $-20p^2q^2 + \frac{1}{4}p - 18 - \frac{1}{3}q$

Remember that when you handwrite a lower-case x in mathematics, you should make it look different from a multiplication sign. One way to do this is to write it as a 'backwards c' followed by a 'normal c', like this:

When we discuss the terms of an expression, we often omit the plus signs. This is convenient in the same way that it's convenient to write the number +3 as 3. So, for example, we might say that the expression

$$-2xy + 3z - y^2 \tag{2}$$

has terms

$$-2xy, \quad 3z \quad \text{and} \quad -y^2.$$

We *never* omit the minus signs! And, of course, we *never* omit the plus sign of a term when writing the term as part of an expression, unless it's the first term.

There's a useful way to think of the relationship between an expression and its terms.

Remember that a *sum* of numbers is what you get by adding them. For example, the sum of 1, 4 and 7 is $1 + 4 + 7 = 12$.

> An expression is equivalent to the sum of its terms.

For example, here is expression (2) written as the sum of its terms:

$$-2xy + 3z - y^2 = -2xy + 3z + (-y^2).$$

The expression on the right is obtained by adding the terms of expression (2) on the left. The two expressions are equivalent because subtracting y^2 is the same as adding the negative of y^2.

You saw in Unit 1 that subtracting a number is the same as adding its negative. For example, $1 - 3$ is the same as $1 + (-3)$.

You saw another example of an expression written as the sum of its terms in Activity 7(g):

$$a - b - 2c = a + (-b) + (-2c).$$

Because the order in which numbers are added doesn't matter, you can change the order of the terms in an expression however you like, and you will obtain an equivalent expression, as long as you keep each term together with its sign. For example, you can swap the order of the first two terms in the expression

$$-2xy + 3z - y^2$$

to give

$$3z - 2xy - y^2.$$

Or you can reverse the original order of the terms to give

$$-y^2 + 3z - 2xy.$$

All three of these expressions are equivalent to each other.

When you do the next activity, you'll probably find it helpful to begin by marking the terms in the way shown on page 14, including their signs, of course. Then think of moving the marked terms around.

Remember that you may need to write a plus sign in front of the first term.

Activity 9 *Changing the order of terms*

Write each of the following expressions with its terms in reverse order.

(a) $-X + 20Y - 5Z$ (b) $2u - 3uv$ (c) $4i - j + 5$

(d) $a - b + c + d$

Changing the order of the terms doesn't simplify an expression, but some methods for simplifying expressions are easier to apply if you rearrange the terms first.

A term in an expression may be just a number, like 4, $\frac{1}{2}$ or -5. If so, we say that it's a **constant term**, or just a *constant* for short. For example, the expression

$$3pq - 2 + 5p^2$$

has one constant term, -2.

It's called a constant term because, unlike other terms, its value doesn't change when the values of the letters in the expression are changed.

On the other hand, if a term is of the form

a number $\times$ a combination of letters,

then the number is called the **coefficient** of the term, and we say that the term is a term *in* whatever the combination of letters is. For example,

$2xy$ has coefficient 2 and is a term in xy;
$-3z$ has coefficient -3 and is a term in z;
$\frac{2}{3}c^2$ has coefficient $\frac{2}{3}$ and is a term in c^2.

The word 'coefficient' was introduced by François Viète (see page 13).

You may be tempted to think that terms like a and $-a$ don't have coefficients. However, because they are equivalent to $1a$ and $-1a$, respectively, they have coefficients 1 and -1, respectively. (We normally write a rather than $1a$, and $-a$ rather than $-1a$, for conciseness.)

Here are some more examples:

y^3 has coefficient 1 and is a term in y^3;
$-ab^2c$ has coefficient -1 and is a term in ab^2c.

Activity 10 *Identifying coefficients*

Write down the coefficient of:

(a) the third term in $2x^2 + 3xy + 4y^2$

(b) the second term in $2\sqrt{p} - 9\sqrt{q} - 7$

(c) the third term in $2x + 5\sqrt{2} + x^2$

(d) the first term in $-a^2b + 2c$

(e) the term in m^2 in $1 + 2m - 3m^2$

(f) the term in b in $ab + 2b + b^2$.

Activity 11 *Indentifying constant terms*

For each of parts (a)–(f) in Activity 10 above, write down any constant terms in the expression.

2.3 Collecting like terms

In this subsection you'll learn the first of several useful techniques for simplifying expressions: *collecting like terms.*

Let's look first at how it works with numbers. If you have 2 batches of 4 dots, and another 3 batches of 4 dots, then altogether you have

2 + 3 batches of 4 dots, that is, 5 batches of 4 dots.

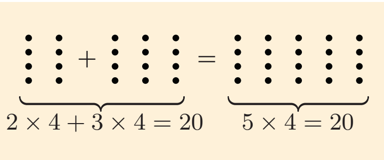


Figure 2

This is shown in Figure 2, and you can express it by writing

$$2 \times 4 + 3 \times 4 = 5 \times 4.$$

Of course, this doesn't work just with batches of four dots. For example, Figure 3 illustrates that

$$2 \times 7 + 3 \times 7 = 5 \times 7.$$

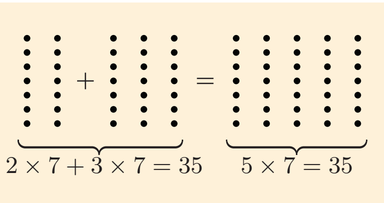


Figure 3

In fact, no matter what number a is,

$$2a + 3a = 5a.$$

This gives you a way to simplify expressions that contain a number of batches of something, added to another number of batches *of the same thing*. For example, consider the expresssion

$$5bc + 4bc.$$

Adding 4 batches of bc to 5 batches of bc gives 9 batches of bc:

$$5bc + 4bc = (5 + 4)bc = 9bc.$$

Terms that are 'batches of the same thing' are called **like terms**. For terms to be like terms, the letters and the powers of the letters in each term must be the same. So, for example,

$7\sqrt{A}$ and $3\sqrt{A}$ are like terms because they are both terms in $\sqrt{A}$;
$2x^2$ and $-0.5x^2$ are like terms because they are both terms in x^2.

However,

$5c$ and $4c^2$ are not like terms because $5c$ is a term in c and $4c^2$ is a term in c^2.

Activity 12 *Identifying like terms*

Which of the following are pairs of like terms?

(a) $3b$ and $6b^2$ (b) $5D$ and $5D$ (c) z and $-z$ (d) 3 and $2m$

Like terms can always be collected in a similar way to the examples above: you just add the coefficients (including negative ones). You can add any number of like terms.

Example 2 *Collecting like terms*

Simplify the following expressions.

(a) $12m + 15m - 26m$ (b) $0.5XY^2 + 0.1XY^2$ (c) $5p - p$

(d) $\frac{1}{3}d - 2d$

Solution

(a) $12m + 15m - 26m = (12 + 15 - 26)m = 1m = m$

(b) $0.5XY^2 + 0.1XY^2 = (0.5 + 0.1)XY^2 = 0.6XY^2$

(c) $5p - p = 5p - 1p = (5 - 1)p = 4p$

(d) $\frac{1}{3}d - 2d = \left(\frac{1}{3} - 2\right) d = \left(\frac{1}{3} - \frac{6}{3}\right) d = -\frac{5}{3}d$

Notice that in the solution to Example 2(d), the fractional coefficients were not converted to approximate decimal values. In algebra you should work with *exact* numbers, such as $\frac{1}{3}$ and $\sqrt{5}$, rather than decimal approximations, wherever possible. However, if you're using algebra to solve a practical problem, then you may have to use decimal approximations.

Activity 13 *Collecting like terms*

Simplify the following expressions.

(a) $8A + 7A$ (b) $-5d + 8d - 2d$ (c) $-7z + z$

(d) $1.4pq + 0.7pq - pq$ (e) $\frac{1}{2}n^2 - \frac{1}{3}n^2$

It's easier to spot like terms if you make sure that all the letters in each term are written in alphabetical order. For example, $5st$ and $2ts$ are like terms – this is easier to see if you write the second one as $2st$.

You can always change the order in which things are multiplied, as this doesn't affect the overall result. For example, $3 \times 4 = 4 \times 3$, and $ts = st$.

Example 3 *Recognising like terms*

Simplify the following expressions.

(a) $5st + 2ts$ (b) $-6q^2rp + 4prq^2$

Solution

(a) $5st + 2ts = 5st + 2st = 7st$

(b) $-6q^2rp + 4prq^2 = -6pq^2r + 4pq^2r = -2pq^2r$

The lower- and upper-case versions of the same letter are *different* symbols in mathematics. So, for example, $4y$ and $9Y$ are not like terms.

Any two constant terms are like terms. They can be collected using the normal rules of arithmetic.

Activity 14 *Recognising like terms*

Which of the following are pairs of like terms?

(a) $2ab$ and $5ab$ (b) $-2rst$ and $20rst$ (c) $2xy$ and $-3yx$

(d) $4c^2a$ and $9ac^2$ (e) abc and cba (f) $8c^2d$ and $9d^2c$

(g) $2A^2$ and $10a^2$ (h) $3fh$ and $3gh$ (i) 22 and -81

Often an expression contains some like terms and some unlike terms. You can simplify the expression by first changing the order of its terms so that the like terms are grouped together, and then collecting the like terms. This leaves an expression in which all the terms are unlike, which can't be simplified any further. Here's an example.

Example 4 *Collecting more like terms*

Simplify the following expressions.

(a) $2a + 5b - 7a + 3b$ (b) $12 - 4pq - 2q + 1 - qp - 2$

Solution

(a) Group the like terms, then collect them.

$$\begin{aligned} 2a + 5b - 7a + 3b &= 2a - 7a + 5b + 3b \\ &= -5a + 8b \end{aligned}$$

(b) Write qp as pq, group the like terms, then collect them.

$$\begin{aligned} 12 - 4pq - 2q + 1 - qp - 2 &= 12 - 4pq - 2q + 1 - pq - 2 \\ &= 12 + 1 - 2 - 4pq - pq - 2q \\ &= 11 - 5pq - 2q \end{aligned}$$

The terms in the final expression can be written in any order. For example, an alternative answer is $11 - 2q - 5pq$.

Activity 15 *Collecting more like terms*

You may find it helpful to mark the terms before rearranging them.

Simplify the following expressions.

(a) $4A - 3B + 3C + 5A + 2B - A$ (b) $-8v + 7 - 5w - 2v - 8$

(c) $20y^2 + 10xy - 10y^2 - 5y - 5xy$ (d) $-4ef + 8e^2f + 10fe - 3f^2e$

(e) $\frac{1}{2}a + \frac{1}{3}b + 2a + \frac{1}{4}b$

As you become more used to working with expressions, you'll probably find that you can collect like terms without grouping them together first. The worked examples in the module will usually do this, and you should do so too, as soon as you feel comfortable with it.

Sometimes when you collect two or more like terms, you find that the result is zero – that is, the terms **cancel** each other out. Here's an example.

Example 5 *Terms that cancel out*

Simplify the expression

$$M + 2N + 3M - 2N.$$

Solution

$$M + 2N + 3M - 2N = 4M + 0 = 4M$$

In the example above, $2N$ is added and then subtracted, and the addition and subtraction cancel each other out.

Activity 16 *Terms that cancel out*

Simplify the following expressions.

(a) $2a^3 - 3a - 2a^3 - 3a$ (b) $2m + n - 5m + 2n + 3m$

(c) $b + 2b + 3b - 6b$

Earlier in the unit, the formula

$$P = 1.24n - 0.69n$$

was found for a baker's profit, where P is the profit in £, and n is the number of loaves. Then the simpler formula

$$P = 0.55n$$

was found for the profit, by thinking about the situation in a different way. Notice that the simpler formula could have been obtained directly from the first formula, by collecting the like terms on the right-hand side:

$$P = 1.24n - 0.69n = 0.55n.$$

In the next activity there is another formula that can be simplified in this way.

Activity 17 *Collecting like terms in a formula*

A primary school parents' group is organising an outing to a children's activity centre, for c children and a adults, travelling by train.

(a) The cost of a ticket for the activity centre is £10 for a child and £2 for an adult. Find a formula for A, where £A is the total cost of admission to the activity centre for the group.

(b) The cost of a return train ticket to get to the activity centre is £7 for a child and £14 for an adult. Find a formula for T, where £T is the total cost of travel for the group.

(c) By adding your answers to parts (a) and (b), find a formula for C, where £C is the total cost of the trip for the group.

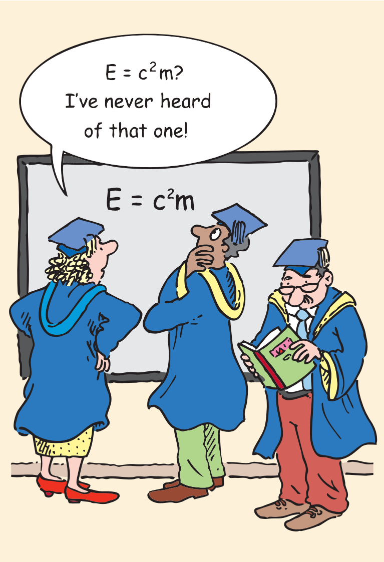
(d) Simplify the formula found in part (c) by collecting like terms, if you haven't already done so in part (c).

(e) Use the formula found in part (d) to find the cost of the trip for 22 children and 10 adults.

Remember that you should not include units, such as £, in a formula. However, if a quantity found using a formula has units, then these should be included in the answer. For example, the answer to part (e) of this activity should include £.

In this section you have been introduced to some terminology used in algebra, and you have learned the algebraic technique of collecting like terms. In the next two sections you'll learn some more algebraic techniques that you will often need to use.

3 Simplifying terms



Sometimes the terms in an expression need to be simplified, to make the expression easier to work with, and to make it easy to recognise any like terms. You'll learn how to do that in this section. We begin by looking at terms individually, and later in the section we consider expressions with more than one term.

3.1 Simplifying single terms

As you've seen, if a term consists of numbers and letters all multiplied together, then it should be written with the coefficient first, followed by the letters. It's often useful to write the letters in alphabetical order – for example, $3B^2DA$ as $3AB^2D$ – as this can help you to identify like terms in a complicated expression. This is usually done in this module. (However, there are some contexts where tradition requires a non-alphabetical order.)

If a term includes a letter multiplied by itself, then index notation should be used. For example,

Remember that the 2 in p^2 and the 3 in p^3 are called *powers*, *indices* (the singular is *index*) or *exponents*. As you saw in Unit 3, we also call p^2 and p^3 *powers* of p, so the word 'power' has two different, but related, meanings in mathematics.

$$p \times p \quad \text{should be simplified to} \quad p^2,$$

and

$$p \times p \times p \quad \text{should be simplified to} \quad p^3.$$

Example 6 *Simplifying terms*

Write the following terms in their shortest forms.

(a) $3 \times c \times g \times 4 \times b$ (b) $b \times a \times 5 \times b \times b$

Solution

(a) $3 \times c \times g \times 4 \times b = 12bcg$

(b) $b \times a \times 5 \times b \times b = 5ab^3$

When you simplify a term you should normally use index notation only for letters and not for numbers. For example,

$$3 \times 3 \times a \quad \text{should be simplified to} \quad 9a, \quad \text{not} \quad 3^2a.$$

Activity 18 *Using index notation*

Write the following terms in their shortest forms.

(a) $y \times z \times 6 \times x \times 4$ (b) $7p \times 2qr$ (c) $QR \times G \times 5F$

(d) $2 \times a \times a \times 3 \times a$ (e) $m \times n \times m \times 4$

(f) $5y \times 2yx$ (g) $4AB \times 4AB$

If a term contains more than one power of the same letter, multiplied together, then the indices need to be *added*. For example,

$$x^3 \times x^2 = x^5.$$

In Unit 3 you met the index law
$$a^m \times a^n = a^{m+n}.$$

Remember that x is the same as x^1; for example,

$$x \times x^7 = x^8.$$

Example 7 *Multiplying powers*

Write the following term in its shortest form:

$$2A^5B \times 3A^4B^7.$$

Solution

$$2A^5B \times 3A^4B^7 = 2 \times 3 \times A^{5+4}B^{1+7} = 6A^9B^8$$

Activity 19 *Multiplying powers*

Write the following terms in their shortest forms.

(a) $8P^8 \times 5P$ (b) $2c^{10}d^3 \times 2c^2d^3$

If a term consists of numbers and letters multiplied together, and some of these have minus signs, then the overall sign of the term can be worked out using the rules below. The rest of the term can be simplified in the usual way.

When multiplying or dividing:

two signs the same give a plus sign;

two different signs give a minus sign.

Similar rules were given for numbers in Unit 1, Subsection 3.1. The following table might help you to remember them.

	+	−
+	+	−
−	−	+

It indicates that a positive times a positive is a positive, a positive times a negative is a negative, and so on.

Here are some examples to illustrate these rules.

$$2 \times (-3) = -6, \qquad (-2) \times (-3) = 6,$$
$$a \times (-b) = -ab, \qquad (-a) \times (-b) = ab.$$

Example 8 *Simplifying terms involving minus signs*

Write the following terms in their shortest forms.

(a) $4q \times (-2p)$ (b) $-B^3 \times (-5B)$ (c) $-a \times (-b) \times (-a)$

Solution

(a) A positive times a negative gives a negative.

$$\begin{aligned} 4q \times (-2p) &= -4q \times 2p \\ &= -8pq \end{aligned}$$

Instead of 'a positive times a negative gives a negative', we sometimes say, informally, 'a plus times a minus gives a minus'.

(b) A negative times a negative gives a positive.

$$\begin{aligned} -B^3 \times (-5B) &= +B^3 \times 5B \\ &= +5B^4 \\ &= 5B^4 \end{aligned}$$

(c) The first negative times the second negative gives a positive, then that positive times the third negative gives a negative.

The overall sign is found in the same way as when you multiply several negative numbers together. See Unit 1, Subsection 3.1.

$$\begin{aligned} -a \times (-b) \times (-a) &= -a \times b \times a \\ &= -a^2b \end{aligned}$$

The box below summarises how to simplify a term.

Strategy *To simplify a term*

1. Find the overall sign and write it at the front.
2. Simplify the rest of the coefficient and write it next.
3. Write the letters in alphabetical order (usually), using index notation as appropriate.

If the coefficient of a term is 1 or -1, then the 1 should be omitted. For example,

$$1xy \quad \text{should be simplified to} \quad xy,$$

and

$$-1c^2 \quad \text{should be simplified to} \quad -c^2.$$

In the next activity, try to simplify the terms in a single step, using the strategy above. This skill will be helpful later, when you learn to simplify more complicated expressions.

Activity 20 *Simplifying terms involving minus signs*

Write the following terms in their shortest forms.

(a) $9X \times (-XY)$ (b) $3s \times \frac{1}{3}r$ (c) $-3a^3 \times (-4a^4)$

(d) $-2pq \times (-3qp^2)$ (e) $-0.5g \times 2f^5$ (f) $-a \times b \times (-c) \times (-d)$

(g) $(-x) \times (-y) \times (-x^2) \times (-4y)$ (h) $(-3cd)^2$ (i) $-(3cd)^2$

Hint for parts (h) and (i): something^2 = the something × the something.

Expressions can contain terms of the form

$$+\,(-\,\text{something}) \quad \text{or} \quad -\,(-\,\text{something}).$$

These should be simplified by using the following facts.

You saw these rules for numbers in Unit 1, Subsection 3.1.

- Adding the negative of something is the same as subtracting the something.
- Subtracting the negative of something is the same as adding the something.

Here are some examples:

$$+(-8) = -8, \qquad -(-5) = +5 = 5,$$
$$+(-2M^2) = -2M^2, \qquad -(-x) = +x = x.$$

Also, any unnecessary brackets in a term should usually be removed. For example:

$$+(pq) = +pq = pq,$$
$$-(7z) = -7z.$$

The brackets in these terms aren't needed, because by the BIDMAS rules multiplication is done before addition and subtraction.

Before removing unnecessary brackets, check carefully that they really are unnecessary! If you're not sure, leave the brackets in.

Activity 21 *Simplifying the signs of terms*

Write the following terms in their shortest forms.

(a) $+(-ab)$ (b) $-(-6x^2)$ (c) $-(2M^4)$ (d) $+(-7y)$

(e) $+(5p)$ (f) $-\left(-\frac{3}{4}n\right)$

3.2 Simplifying two or more terms

So far in this section you've been simplifying single terms. To simplify an expression with two or more terms, you need to simplify each term individually, in the way that you've seen. Before you can do that, you need to identify which bits of the expression belong to which term. The easiest way to do that is to use the following fact.

> Each term after the first starts with a plus or minus sign that isn't inside brackets.

Tutorial clip

Example 9 *Identifying terms*

Mark the terms in the following expressions.

(a) $-2a - (-5a^2) + (-4a)$ (b) $2x \times 4xy - 2y \times (-5x)$

Solution

(a) Begin by marking the start of the first term.

$$\underline{-}2a - (-5a^2) + (-4a)$$

Extend the line under the first term until you reach a plus or minus sign that isn't inside brackets. That's the start of the next term.

$$\underline{-2a}\ \underline{-}(-5a^2) + (-4a)$$

Extend the line under the second term until you reach a plus or minus sign that isn't inside brackets. That's the start of the next term.

$$\underline{-2a}\ \underline{-(-5a^2)}\ \underline{+}(-4a)$$

Extend the line under the third term until you reach a plus or minus sign that isn't inside brackets. This time you don't reach one –

you just reach the end of the expression. So this expression has three terms, as marked.

$$\underbrace{-2a}\ \underbrace{-(-5a^2)}\ \underbrace{+(-4a)}$$

(b) Mark the start of the first term.

$$\underbrace{2}x \times 4xy - 2y \times (-5x)$$

When you reach a plus or minus sign that isn't inside brackets, that's the start of the next term.

$$\underbrace{2x \times 4xy}\ \underbrace{-}\ 2y \times (-5x)$$

You don't reach another plus or minus sign that isn't inside brackets, so this expression has two terms.

$$\underbrace{2x \times 4xy}\ \underbrace{-2y \times (-5x)}$$

Example 9 showed you how to carry out the first step in the following strategy. The other two steps use techniques that you've seen already.

Strategy *To simplify an expression with more than one term*

1. Identify the terms. Each term after the first starts with a plus or minus sign that isn't inside brackets.
2. Simplify each term, using the strategy on page 22. Include the sign (plus or minus) at the start of each term.
3. Collect any like terms.

As usual, a plus sign in front of a first term can be omitted.

In the next example, this strategy is used to simplify the expressions in Example 9.

Tutorial clip

Example 10 *Simplifying expressions with more than one term*

Simplify the following expressions.

(a) $-2a - (-5a^2) + (-4a)$ (b) $2x \times 4xy - 2y \times (-5x)$

Solution

(a) First identify the terms. Then simplify each term individually. Finally, collect like terms.

$$\begin{aligned}\underbrace{-2a}\ \underbrace{-(-5a^2)}\ \underbrace{+(-4a)} &= -2a + 5a^2 - 4a \\ &= 5a^2 - 6a\end{aligned}$$

This could also be written as $-6a + 5a^2$.

(b) Identify the terms, then simplify each term individually. Finally, check for like terms – there are none here.

$$\underbrace{2x \times 4xy}\ \underbrace{-2y \times (-5x)} = 8x^2y + 10xy$$

In the next activity, begin each part by marking the terms, as shown in Example 9. As you become more used to manipulating expressions, you'll probably find that you can identify and simplify the terms without needing to mark them.

Activity 22 *Simplifying expressions*

Simplify the following expressions.

(a) $5m \times 2m - 2n \times n^2$ (b) $3p \times 2q + 2r \times (-7p)$

(c) $2P - (-3Q) + (-P) + (2Q)$ (d) $3 \times (-2a) - 1c^2 + 9ac$

(e) $4s \times \frac{1}{2}rst - 2(-\frac{1}{2}s)$ (f) $-5xy + (-3y \times x^2) - (-y^2)$

(g) $-3r \times (-2r) - (-2r \times r) + (r^2 \times 9)$

Don't be concerned if you find this activity difficult – it's one of the harder ones in the unit. Take your time, and follow the strategy carefully. Remember that some terms may not need to be simplified, as they may already be in their simplest forms.

4 Brackets

In this section you'll learn how to rewrite expressions that contain brackets as expressions without brackets, and you'll also see some applications of algebra, including how to prove that the number trick in Subsection 1.1 always works.

4.1 Multiplying out brackets

Any expression that contains brackets, such as

$$8a + 3b(b - 2a)$$

or

$$(2m + 3n) - (m + n - 3r),$$

can be rewritten without brackets. To see how to do this, let's start by looking at an expression that involves only numbers:

$$(2 + 3) \times 4.$$

When you learned to collect like terms, you saw that 2 + 3 batches of 4 dots is the same as 2 batches of 4 dots plus 3 batches of 4 dots, as illustrated in Figure 4. So $(2 + 3) \times 4$ is equivalent to

$$2 \times 4 + 3 \times 4.$$

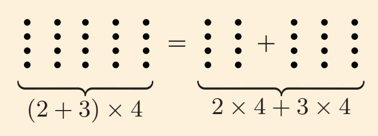


Figure 4

Here an expression containing brackets has been rewritten as an expression without brackets:

$$(2 + 3) \times 4 = 2 \times 4 + 3 \times 4.$$

It's usual to write numbers in front of brackets, so let's write the 4 first in each multiplication:

$$4(2 + 3) = 4 \times 2 + 4 \times 3.$$

Here you can see how to rewrite an expression with brackets as one without brackets: you multiply each of the numbers inside the brackets individually by the number outside the brackets. This is called **multiplying out the brackets**, *expanding the brackets*, or simply *removing the brackets*. The number outside the brackets is called the **multiplier**. Here's another example, with multiplier 7.

$$7(1 + 5) = 7 \times 1 + 7 \times 5$$

Multiplying out the brackets can be particularly helpful for expressions that contain letters. The rule above applies in just the same way.

Strategy *To multiply out brackets*

Multiply each term inside the brackets by the multiplier.

Remember that you must multiply *every* term inside the brackets, not just the first term.

Here are two examples, with multipliers a and 3, respectively.

$$a(b+c) = ab + ac$$

$$3(p+q^2+r) = 3p + 3q^2 + 3r$$

It doesn't matter whether the multiplier is before or after the brackets. Here's an example of multiplying out where the multiplier is after the brackets:

$$(x+y)z = xz + yz.$$

If you prefer the multiplier to be before the brackets, then you can change the order before multiplying out. For example,

$$(x+y)z = z(x+y) = zx + zy = xz + yz.$$

When you multiply out brackets, you often need to simplify the resulting terms, as illustrated in the next example.

Tutorial clip

Example 11 *Multiplying out brackets*

Multiply out the brackets in the following expression:

$$2a(3a+2b).$$

Solution

$$\begin{aligned} 2a(3a+2b) &= 2a \times 3a + 2a \times 2b \\ &= 6a^2 + 4ab \end{aligned}$$

Activity 23 *Multiplying out brackets*

Multiply out the brackets in the following expressions.

(a) $3p(pq+4)$ (b) $x^2(x+2y)$

Once you're familiar with how to multiply out brackets, it's usually best to simplify the terms as you multiply out, instead of first writing down an expression containing multiplication signs. This leads to tidier expressions and fewer errors.

For example, if you look at the expression in Example 11,

$$2a(3a+2b),$$

you can see that when you multiply out the brackets, the first term will be $2a$ times $3a$. You simplify this to $6a^2$, using the strategy of first finding the sign, then the rest of the coefficient and then the letters, and write it

down. Then you see that the second term is $2a$ times $+2b$, simplify this to $+4ab$, and write it down after the first term. This gives

$$2a(3a + 2b) = 6a^2 + 4ab.$$

Try this shorter form of working in the following activity.

Activity 24 *Multiplying out brackets efficiently*

Multiply out the brackets in each of the following expressions.

(a) $f(e + 5g)$ (b) $5(2A + B)$ (c) $3c(4c + 2d)$

(d) $(a - b)c^2$ (e) $2y(x + 2y + 4z)$ (f) $2\left(\frac{1}{2}A^2 + \frac{3}{2}\right)$

(g) $a(x + y)z$ (h) $2b(b^2 + 2b^4)$

Simplifying the terms at the same time as multiplying out is particularly helpful when some of the terms inside the brackets, or the multiplier, have minus signs. For example, let's multiply out the brackets in the expression

$$3m(-2m + 3n - 6).$$

The first term is $3m$ times $-2m$, which simplifies to $-6m^2$. Working out the other terms in a similar way, we obtain

$$3m(-2m + 3n - 6) = -6m^2 + 9mn - 18m.$$

Here's another example. In this one the multiplier has a minus sign.

Example 12 *Multiplying out brackets involving minus signs*

Tutorial clip

Multiply out the brackets in the following expression:

$$-a(b - a + 7).$$

Solution

$$-a(b - a + 7) = -ab + a^2 - 7a$$

The terms are
$-a \times b = -ab,$
$-a \times (-a) = +a^2,$
$-a \times 7 = -7a.$

Activity 25 *Multiplying out brackets involving minus signs*

Multiply out the brackets in the following expressions, simplifying where possible.

(a) $p(q - r)$ (b) $7a(-4a + 3b)$ (c) $6(0.2a - 0.3b + 1.4)$

(d) $10(\frac{1}{2}n + \frac{1}{5})$ (e) $-3(x - 2y)$ (f) $-b^2(-a + b)$

An expression containing brackets may have more than one term. For example, the expression

$$x(y + 1) + 2y(y + 3)$$

has two terms, each containing brackets, as follows:

$$\underbrace{x(y + 1)}\ \underbrace{+\,2y(y + 3)}.$$

The strategy for simplifying expressions is on page 24.

An expression like this can be dealt with term by term, using a similar strategy to the one for simplifying expressions.

Strategy *To multiply out brackets in an expression with more than one term*

1. Identify the terms. Each term after the first starts with a plus or minus sign that isn't inside brackets.
2. Multiply out the brackets in each term. Include the sign (plus or minus) at the start of each resulting term.
3. Collect any like terms.

Stage 2 usually increases the number of terms.

While you're learning to multiply out brackets in expressions with more than one term, you'll probably find it helpful to mark the terms as you identify them. This should help you to avoid errors.

Tutorial clip

Example 13 *Expanding the brackets when there's more than one term*

Multiply out the brackets in the following expressions, simplifying where possible.

(a) $x(y+1)+2y(y+3)$ (b) $2r^2-r(r-s)$

Solution

(a) Identify the terms. Multiply out the brackets. Then check for like terms – there are none here.

$$\underbrace{x(y+1)}\ \underbrace{+\,2y(y+3)} = xy + x + 2y^2 + 6y$$

(b) Identify the terms. Multiply out the brackets. Collect like terms.

$$\begin{aligned}\underbrace{2r^2}\ \underbrace{-\,r(r-s)} &= 2r^2 - r^2 + rs\\ &= r^2 + rs\end{aligned}$$

Activity 26 *Expanding the brackets when there's more than one term*

Multiply out the brackets in the following expressions, simplifying where possible.

(a) $f+g(f+h)$ (b) $x-y(x+2y)$ (c) $2p-3q(-3p+q)$

(d) $-2(a+b)+4(a-b)$ (e) $2aE-3E(-E-5a)$ (f) $(d-c)c-c^2$

Some expressions, such as

$$-(a+2b-c),$$

contain brackets with just a minus sign in front.

You can remove these brackets by using the fact that a minus sign in front is just the same as multiplying by -1:

$$\begin{aligned}-(a+2b-c) &= -1(a+2b-c)\\ &= -a-2b+c.\end{aligned}$$

You can see that the overall effect is that the sign of each term in the brackets has been changed.

An expression may also contain brackets with just a plus sign in front. These brackets can be removed by using the fact that a plus sign in front is just the same as multiplying by 1. For example, the expression

$$2x + (y - 3z)$$

can be simplified as follows:

$$\begin{aligned} \underbrace{2x}\ \underbrace{+\,(y - 3z)} &= \underbrace{2x}\ \underbrace{+\,1(y - 3z)} \\ &= 2x + y - 3z. \end{aligned}$$

This time you can see that the effect is that all the signs in the brackets remain as they are.

Rather than introducing 1 or -1 into working, as was done above, it's better to remember the following strategy.

This strategy applies even if there's just one term in the brackets. For example,

$$+(-2M^2) = -2M^2,$$

and

$$-(-x) = +x = x,$$

as you know already from the rules for adding and subtracting negatives, which were given on page 22.

Strategy *To remove brackets with a plus or minus sign in front*

- If the sign is plus, keep the sign of each term inside the brackets the same.
- If the sign is minus, change the sign of each term inside the brackets.

Tutorial clip

Example 14 *Plus and minus signs in front of brackets*

Remove the brackets in the following expressions.

(a) $-(-P^2 + 2Q - 3R)$ (b) $a + (2bc - d)$

Solution

(a) $$\begin{aligned} \underbrace{-(-P^2 + 2Q - 3R)} &= +P^2 - 2Q + 3R \\ &= P^2 - 2Q + 3R \end{aligned}$$

(b) $\underbrace{a}\ \underbrace{+\,(2bc - d)} = a + 2bc - d$

Activity 27 *Plus and minus signs in front of brackets*

Remove the brackets in each of the following expressions, and simplify them where possible.

(a) $-(4f - g^3)$ (b) $-(-x + 7y - 8z + 6)$ (c) $2(a - b) + (c - 2d)$

(d) $r + (-2s - r)$ (e) $-A + B - (-3A + 4B)$

(f) $-(-t - w) + (-t + w)$ (g) $-(L + 2M) - (-M)$

Some expressions, such as

$$(x + 2)(x - 5),$$

contain two, or even more, pairs of brackets multiplied together. You'll learn how to multiply out brackets like these in Unit 9.

You've seen that you should usually write expressions in the simplest way you can. For example, you should write

$$x + 2x + 3x \quad \text{as} \quad 6x.$$

The second form of this expression is clearly simpler than the first:

- it's shorter and easier to understand, and
- it's easier to evaluate for any particular value of x.

These are the attributes to aim for when you try to write an expression in a simpler way.

However, sometimes it's not so clear that one way of writing an expression is better than another. For example,

$$x(x + 1) \quad \text{is equivalent to} \quad x^2 + x.$$

Both these forms are reasonably short, and both are reasonably easy to evaluate. So this expression doesn't have a simplest form.

The same is true of many other expressions. You should try to write each expression that you work with in a reasonably simple way, but often there's no 'right answer' for the simplest form. One form might be better for some purposes, and a different form might be better for other purposes.

In particular, multiplying out the brackets in an expression doesn't always simplify it.

In Unit 7 you'll see that there's a reverse process to multiplying out the brackets: sometimes an expression can be made simpler or more useful by *introducing* brackets.

4.2 Algebraic fractions

It's usually best to use fraction notation, rather than a division sign, to indicate division in algebraic expressions. For example, the expression

$$a + b \div c \tag{3}$$

can be written as

$$a + \frac{b}{c}. \tag{4}$$

This makes it easy to see which parts of the expression are divided by which. In expression (3), it's just b, not $a + b$, that's divided by c, by the BIDMAS rules. This is clearer in expression (4).

Similarly, the expression

$$(8a + 3) \div (2a)$$

can be written in fraction notation as

$$\frac{8a + 3}{2a}. \tag{5}$$

The brackets can be omitted because the fraction notation makes it clear that the whole of $8a + 3$ is divided by $2a$.

Algebraic expressions written using fraction notation are called **algebraic fractions**. The expressions above and below the line in an algebraic fraction are called the *numerator* and *denominator*, respectively, just as they are for ordinary fractions.

When you write an algebraic fraction, you must make sure that the horizontal line extends to the full width of the numerator or denominator, whichever is the wider. For example,

$$(8a + 3) \div (2a) \text{ should be written as } \frac{8a + 3}{2a}, \text{ not } \frac{8a + 3}{\,2a\,}.$$

This is because the line acts as brackets for the numerator and denominator, as well as indicating division.

Try not to use division signs in expressions, or whenever you carry out algebraic manipulation, from now on; use fraction notation instead. However, occasionally it's useful to use division signs in algebraic expressions, just as occasionally it's useful to use multiplication signs.

If you need to type an algebraic fraction in a line of text, for example when sending an email, then use brackets and a slash. For example,

$$\frac{8a+3}{2a}$$

can be typed as $(8a+3)/(2a)$.

Activity 28 *Using fraction notation*

Rewrite the following expressions using fraction notation.

(a) $(a+b) \div 3$ (b) $a+b \div 3$ (c) $(x+2) \div (y+3)$

(d) $(x+2) \div y+3$ (e) $x+2 \div (y+3)$ (f) $x+2 \div y+3$

Activity 29 *Working with fraction notation*

Multiply out the brackets in the following expressions.

(a) $6\left(1+\dfrac{h}{2}\right)$ (b) $6\left(\dfrac{1+h}{2}\right)$

Hint for part (b): First write the expression in the form 'number $\times (1+h)$'.

There's a technique, based on multiplying out brackets, that can be useful when you're working with algebraic fractions. As with multiplying out brackets, this technique doesn't necessarily simplify an expression; it just gives a different way of writing it. It applies to algebraic fractions where there's more than one term in the *numerator*, such as

$$\frac{2a-5b+c}{3d}. \tag{6}$$

Since dividing by something is the same as multiplying by its reciprocal, you can write this expression as

$$\frac{1}{3d}(2a-5b+c).$$

You can then multiply out the brackets to give

$$\frac{2a}{3d}-\frac{5b}{3d}+\frac{c}{3d}. \tag{7}$$

If you compare expressions (6) and (7), you can see that the overall effect is that each term on the numerator has been individually divided by the denominator. This is called **expanding** the algebraic fraction.

Once an algebraic fraction has been expanded, it may be possible to simplify some of the resulting terms, as illustrated in the next example.

Example 15 *Expanding an algebraic fraction*

Expand the algebraic fraction $\dfrac{10x+x^2-8}{x}$.

Solution

$$\begin{aligned}\frac{10x+x^2-8}{x} &= \frac{10x}{x}+\frac{x^2}{x}-\frac{8}{x}\\ &= 10+x-\frac{8}{x}\end{aligned}$$

Remember: $\dfrac{x^2}{x}=\dfrac{\cancel{x}\times x}{\cancel{x}}=x.$

Remember that an algebraic fraction can be expanded only if it has more than one term in the *numerator*. The following fraction, which has more than one term in the denominator, can't be expanded:

$$\frac{a}{2a+5b-c}.$$

Activity 30 *Expanding algebraic fractions*

Expand the following algebraic fractions, and simplify the resulting expressions where possible.

(a) $\dfrac{A-6B}{3}$ (b) $\dfrac{10z^2+5z-20}{5}$ (c) $\dfrac{3A^2+A}{A}$

You'll learn more about working with algebraic fractions in Unit 9.

4.3 Using algebra

You've now covered all the algebra needed to prove that the number trick in Subsection 1.1 works for every possible starting number.

To do this, you carry out the trick starting with a letter (n, say), which represents any number at all. After each step you simplify the resulting expression.

Think of a number:	n
Double it:	$2n$
Add 7:	$2n+7$
Double the result:	$2(2n+7)=4n+14$
Add 6:	$4n+14+6=4n+20$
Divide by 4:	$\dfrac{4n+20}{4}=\dfrac{4n}{4}+\dfrac{20}{4}=n+5$
Take away the number you first thought of:	$n+5-n=5$

So, because n represents any number at all, you can see that you'll always end up with the answer 5.

Notice that it was important to include the brackets when the expression $2n+7$ was doubled in the above calculation. Doubling $2n+7$ does *not* give $2\times 2n+7=4n+7$; the correct calculation is

$$2(2n+7)=4n+14.$$

Activity 31 *Checking a number trick*

Here's another number trick.

Think of a number.
Multiply it by 3.
Add 2.
Double the result.
Add 2.
Divide by 6.
Take away the number you first thought of.

(a) Test the trick with whatever starting number you wish. What's the final answer?

(b) By starting with n, prove that the trick always gives the same answer.

In the next example you'll see how multiplying out brackets can be useful when you're finding a formula.

Example 16 *Finding and simplifying a formula*

Each week Arthur works for at least 37 hours. He's paid £15 per hour for the first 37 hours, and £25 per hour for each additional hour. Find a formula for P, where £P is Arthur's pay for the week if he works for n hours.

Solution

Arthur's pay in £ for the first 37 hours is

$$37 \times 15 = 555.$$

The number of additional hours that Arthur works is $n - 37$, so his pay in £ for these additional hours is

$$(n - 37) \times 25 = 25(n - 37).$$

The question states that Arthur works for at least 37 hours, so there's no need to worry about $n - 37$ going negative.

So a formula for Arthur's pay is

$$P = 555 + 25(n - 37).$$

The formula can be simplified by multiplying out the brackets:

$$\begin{aligned} P &= 555 + 25n - 925 \\ &= 25n - 370. \end{aligned}$$

So the simplified formula is

$$P = 25n - 370.$$

Activity 32 *Finding and simplifying a formula*

A print company charges £175 to print 200 leaflets of a particular size, plus an extra £0.25 per leaflet for any further leaflets.

(a) Find a formula for C, where £C is the cost of printing n leaflets, assuming that n is at least 200.

Remember that the units for C are pounds. Don't convert any of the costs to pence.

(b) Write the formula as simply as you can.

(c) Use the formula to find the cost of printing 450 leaflets.

Finally in this section, you'll see some more examples of how algebra can be used to prove mathematical facts. First try the following activity.

Activity 33 *Adding three consecutive integers*

Choose any three consecutive integers and add them together. Is the total divisible by 3? Now try another three consecutive integers.

As with the number tricks you saw earlier, the property in Activity 33 seems to hold for any integers you choose. But how can you be sure that it always holds? You can't check *all* choices of three consecutive integers individually, as there are infinitely many choices. The way to prove that the property always holds is to use algebra.

Tutorial clip

Example 17 *Proving a property of numbers*

Prove that the sum of any three consecutive integers is divisible by 3.

Solution

Represent the first of the three integers by n. Then the other two integers are $n+1$ and $n+2$. So their sum is

$$\begin{aligned} n+(n+1)+(n+2) &= n+n+1+n+2 \\ &= 3n+3. \end{aligned}$$

Dividing by 3 gives

$$\begin{aligned} \frac{3n+3}{3} &= \frac{3n}{3}+\frac{3}{3} \\ &= n+1. \end{aligned}$$

Now $n+1$ is an integer, because n is an integer. So dividing the sum of the three numbers by 3 gives an integer. That is, the sum is divisible by 3.

To confirm that $3n+3$ is a multiple of 3, you can either divide by 3 and check that you get an integer, as is done in the example, or you can argue as follows. The number $3n$ is a multiple of 3, since n is an integer, so $3n+3$ must also be a multiple of 3, since adding 3 to a multiple of 3 gives another multiple of 3.

Activity 34 *Proving a property of numbers*

(a) Choose any three integers such that the second and third are each 2 more than the one before (such as 5, 7 and 9, or 20, 22 and 24). Is their sum divisible by 3?

(b) Choose another three integers of the same type, and check whether their sum is divisible by 3.

(c) Prove that the same thing always happens. Write out your proof in a similar way to the proof in Example 17.

You can use similar methods to prove many more properties of numbers. Here are another two proofs for you to try.

Activity 35 *Proving another property of numbers*

Prove that if you add up any four consecutive integers, then the answer is *not* divisible by 4.

The facts that you saw in Example 17 and Activity 34 are particular cases of the more general fact explored in the next activity.

Activity 36 *Proving a more general property of numbers*

In this activity you are asked to prove the following fact.

> If you add up any three integers such that the second and third integers are each the same amount more than the one before, then the answer is divisible by 3.

Do this by following the steps below.

(a) Try a numerical example first – you could use one of the two examples in the margin, or choose your own example. Write down the first integer, the amount by which the second and third integers are more than the one before, and the sum of the three integers. Check that the sum of the three integers is divisible by 3.

(b) Use the letter n to represent the first integer and the letter d to represent the amount by which the second and third integers are more than the one before. Write down expressions for the second and third integers in terms of n and d.

(c) Use your answers to part (b) to find an expression for the sum of the three integers, and hence prove the fact above.

For example, the integers 15, 19 and 23 have this property, since they have the following pattern:

$$15 \xrightarrow{+4} 19 \xrightarrow{+4} 23.$$

Similarly, the integers 107, 117 and 127 have the property, since they have the following pattern:

$$107 \xrightarrow{+10} 117 \xrightarrow{+10} 127.$$

5 Linear equations

In Section 1 you met the idea that some mathematical questions can be answered by solving equations. In this section you'll learn how to solve equations of a particular type – *linear equations in one unknown* – and see how to use them to answer some simple mathematical questions.

You'll soon see that linear equations are common in mathematics and can be extremely useful. Later in the module, you'll see how they can be used to model and solve some more complex real-world problems.

5.1 Solutions of equations

As you saw earlier, an equation consists of two expressions, with an equals sign between them. Here's an example:

$$2(r+3) = 5r - 6. \qquad (8)$$

The expressions to the left and right of the equals sign in an equation are referred to as the *left-hand side* (LHS) and *right-hand side* (RHS), respectively. For equation (8) above,

$$\text{LHS} = 2(r+3) \quad \text{and} \quad \text{RHS} = 5r - 6.$$

This section is about equations that contain an *unknown* – a letter representing a number that you don't know. For example, earlier you saw the equation

$$\tfrac{3}{10}N = 150, \qquad (9)$$

This equation was in Subsection 1.3 (see page 9).

where N is an unknown representing the number of children who applied

to a school. It turned out that $N = 500$, because equation (9) is correct when you substitute 500 for N:

$$\tfrac{3}{10} \times 500 = 150.$$

The process of finding the value of the unknown in an equation is called **solving the equation**, and the value found is called a *solution* of the equation. We also say that this value **satisfies** the equation.

Example 18 *Checking a solution of an equation*

Show that $r = 4$ is a solution of the equation

$$2(r + 3) = 5r - 6.$$

Solution

If $r = 4$, then

$$\text{LHS} = 2(4 + 3) = 2 \times 7 = 14$$

and

$$\text{RHS} = 5 \times 4 - 6 = 20 - 6 = 14.$$

Since LHS = RHS, $r = 4$ is a solution.

When you check whether a number is a solution of an equation, you should set out your working in a similar way to Example 18. Evaluate the left- and right-hand sides *separately*, and check whether each side gives the same answer.

If one side of the equation is a constant term, then you just need to evaluate the other side, and check whether you get the right number. You can set out your working in the way shown in Example 19 below.

Example 19 *Checking a solution of another equation*

Show that $x = -2$ is a solution of the equation

$$\tfrac{1}{2}(x + 8) = 3.$$

Solution

If $x = -2$, then

$$\text{LHS} = \tfrac{1}{2}(-2 + 8) = \tfrac{1}{2} \times 6 = 3 = \text{RHS}.$$

Hence $x = -2$ is a solution.

Activity 37 *Checking solutions of equations*

Determine whether each of the following statements is true.

(a) The equation $4p = 12$ has solution $p = 3$.

(b) The equation $10 - 2A = 1 + A$ has solution $A = 2$.

(c) The equation $4z + 2 = 3(z - 1)$ has solution $z = -5$.

It's possible for an equation to have more than one solution. For example, the equation $a^2 = 4$ has two solutions, $a = 2$ and $a = -2$, because $2^2 = 4$ and $(-2)^2 = 4$. It's also possible for an equation to have no solution at all. For example, the equation $a^2 = -1$ has no solution, because squaring a real number always gives a non-negative answer.

All the equations that you'll be asked to solve in this unit have exactly one solution. They're all of a particular type – **linear equations in one unknown** – and each equation of this type has exactly one solution. The phrase 'in one unknown' means that there's only one unknown in the equation (though it can appear more than once). The word 'linear' means that if the unknown is x, say, then after expanding any brackets or fractions in the equation, each term is either a constant term or a number times x. In particular, the equation doesn't involve powers like x^2, x^3 or x^{-1}, but only x itself. Linear equations are related to straight lines, as you'll see in the next unit.

You'll learn more about different types of equations, and how many solutions they have, as part of your work in the next few units of the module.

It's traditional to use the letter x for a single unknown. You can use any letter, but x is often used in general discussions about equations.

Traditionally, unknowns have been represented by letters from near the end of the alphabet. This convention was introduced by the French philosopher and mathematician René Descartes.

Figure 5 René Descartes (1596–1650)

5.2 How to solve linear equations

The method for solving linear equations that you'll learn in this section is based on a simple idea. Consider the following equation:

$$6 = 6. \tag{10}$$

This equation may seem rather boring, but every equation is much like it! In every equation, the expressions on each side of the equals sign represent *equal numbers*.

If you've got two equal numbers, then you can do the same thing to each of them and you'll still have two equal numbers. For example, you can add 4 to each side of equation (10) to obtain

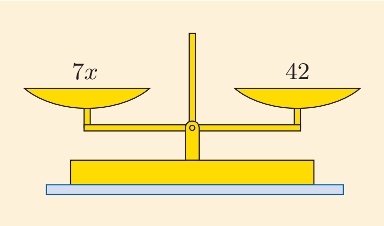
$$6 + 4 = 6 + 4; \quad \text{that is,} \quad 10 = 10.$$

But if you do something to *just one side* of an equation, then things go wrong. For example, if you add 4 to just the left-hand side of equation (10), then you obtain $10 = 6$, which is wrong.

If you do any of the following things to *both sides* of a correct equation, then you obtain another correct equation.

- Add a number.
- Subtract a number.
- Multiply by a number.
- Divide by a non-zero number.

You might find it helpful to think of the two sides of an equation as the two pans of a set of weighing scales. To keep the scales balanced, the same thing must be done to the weights on each side.



This fact can be used to solve equations. Let's use it now to solve the equation

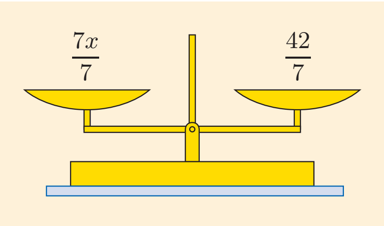
$$7x = 42. \qquad (11)$$

The idea is to do the same thing to both sides and end up with an equation of the form

$$x = \text{a number},$$

since that will give the solution. So what should we do to both sides? Well, since x is *multiplied* by 7 in the equation, we need to *divide* by 7, to give x by itself. Dividing both sides by 7 gives

Multiplying by 7 and dividing by 7 are **inverse** operations – each undoes the effect of the other.



$$\frac{7x}{7} = \frac{42}{7}.$$

Simplifying gives

$$x = 6,$$

which is the solution.

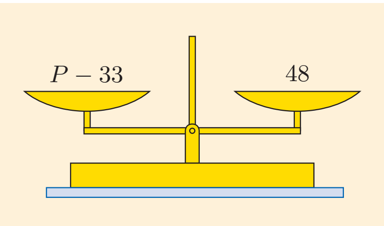
Once you've found the solution to an equation, it's a good idea to check it, to make sure that you haven't made a mistake. For equation (11), if $x = 6$, then

$$\text{LHS} = 7 \times 6 = 42 = \text{RHS}.$$

So the solution found above is correct.

Here's another example, which illustrates how you should set out your working when you solve an equation.

Example 20 *Solving an equation*



Solve the equation $P - 33 = 48$.

Solution

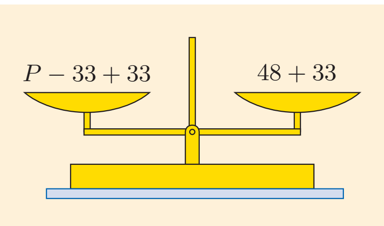
The equation is: $\quad P - 33 = 48$

P has 33 subtracted, so add 33. Add it to both sides.

Adding 33 is the inverse operation to subtracting 33.

Add 33: $\quad P - 33 + 33 = 48 + 33$

Simplify: $\quad P = 81$



The solution is $P = 81$.

(Check: if $P = 81$, then

$$\text{LHS} = 81 - 33 = 48 = \text{RHS},$$

so the solution is correct.)

Activity 38 *Solving equations*

Solve the following equations by doing the same thing to both sides.

(a) $5x = 20$ (b) $t - 6 = 7$ (c) $x + 4 = 1$

(d) $\dfrac{z}{2} = 8$ (e) $x - 1.7 = 3$ (f) $3X = 4$

(g) $-2y = 10$ (h) $\dfrac{c}{-5} = -6$ (i) $-m = 12$

In part (f), remember that it's usually best to give *exact* answers, so use fractions.

You can use the idea of doing the same thing to both sides to solve more complicated equations. Given a complicated equation, you first do the same thing to both sides to obtain a simpler equation. Then you do the same thing to both sides of the simpler equation to obtain an even simpler equation, and so on, until eventually you end up with an equation of the form

$$x = \text{a number},$$

which gives the solution.

The tricky bit is deciding what to do in each step! The best approach is to first aim to obtain an equation of the form

$$\text{a number} \times x = \text{a number},$$

that is, something like $5x = 20$, $3x = -4$ or $-2x = 10$. You can often obtain an equation of this form by adding or subtracting terms on both sides of the original equation. Once you've obtained an equation of this form, you need to carry out just one further step – dividing both sides by the coefficient of x – to obtain the solution. Here's an example.

Example 21 *Solving a more complicated equation*

Tutorial clip

Solve the equation

$$5x = 3x + 10.$$

Solution

The equation is: $5x = 3x + 10$

First aim to get an equation of the form 'a number $\times$ x = a number'. The equation is nearly in this form: the only problem is the $3x$ on the right-hand side. To cancel out the $3x$, subtract $3x$ from both sides.

Subtract $3x$: $5x - 3x = 3x + 10 - 3x$

Since $3x$ represents a number, it's fine to subtract it from both sides.

Simplify: $2x = 10$

This is in the form wanted. To find the solution, divide both sides by 2.

Divide by 2: $\dfrac{2x}{2} = \dfrac{10}{2}$

Simplify: $x = 5$

The solution is $x = 5$.

(Check: if $x = 5$, then

$$\text{LHS} = 5 \times 5 = 25$$

and

$$\text{RHS} = 3 \times 5 + 10 = 15 + 10 = 25.$$

Since LHS = RHS, the solution is correct.)

Activity 39 *Solving more complicated equations*

Solve each of the following equations by first adding or subtracting a term on both sides to obtain an equation of the form 'a number $\times x =$ a number', and then dividing both sides by the coefficient of x. Set out your working in the way shown in Examples 20 and 21.

(a) $9x = 12 + 5x$ (b) $6x + 8 = 2$ (c) $9x = 6 - 3x$

When you're solving an equation, you may need to first add or subtract one term on both sides, and then add or subtract another term on both sides, to obtain an equation of the form 'a number $\times x =$ a number'. This is illustrated in the next example.

Tutorial clip

Example 22 *Solving an even more complicated equation*

Solve the equation

$$7x - 4 = 2x - 14.$$

Solution

The equation is: $7x - 4 = 2x - 14$

For the form 'a number $\times x =$ a number', there must be one term in x, on the left-hand side. To cancel out the $2x$ on the right-hand side, subtract $2x$ from both sides.

Subtract $2x$: $7x - 4 - 2x = 2x - 14 - 2x$

Simplify: $5x - 4 = -14$

For the form 'a number $\times x =$ a number', there must be one constant term, on the right-hand side. To cancel out the -4 on the left-hand side, add 4 to both sides.

Add 4: $5x - 4 + 4 = -14 + 4$

Simplify: $5x = -10$

This is in the form wanted. To find the solution, divide both sides by 5.

Divide by 5: $\dfrac{5x}{5} = \dfrac{-10}{5}$

Simplify: $x = -2$

The solution is $x = -2$.

(Check: if $x = -2$, then

$$\text{LHS} = 7(-2) - 4 = -14 - 4 = -18$$

and

$$\text{RHS} = 2(-2) - 14 = -4 - 14 = -18.$$

Since LHS = RHS, the solution is correct.)

Activity 40 *Solving even more complicated equations*

Solve each of the following equations using the method illustrated in Example 22. That is, first add or subtract a term on both sides, and then add or subtract another term on both sides, to obtain an equation of the form 'a number $\times x =$ a number'. Then divide both sides by the coefficient of x. Set out your working in the way shown in Example 22.

(a) $3x + 2 = x + 10$ (b) $5x + 9 = -x - 3$

Sometimes when you're solving an equation, you can make the working easier by swapping the sides. For example, suppose that you want to solve the equation

$$x + 16 = 5x.$$

You could begin by swapping the sides, because that gives an equation that's closer to the form 'a number $\times x =$ a number':

$$5x = x + 16.$$

(Then you just need to subtract x from both sides to obtain the form you want.)

You can swap the sides of an equation at any stage of your working.

An alternative to swapping the sides is to aim to get a term in x on the *right*-hand side only, and a constant term on the *left*-hand side only, instead of the other way round. Then you end up with a final equation of the form

$$\text{a number} = x$$

instead of the usual

$$x = \text{a number}.$$

Here's a summary of the method that you've seen for solving equations. It can be used for any linear equation that doesn't contain fractions or brackets.

Strategy *To solve a linear equation in one unknown with no fractions or brackets*

Carry out a sequence of steps. In each step, do one of the following:

- do the same thing to both sides
- simplify one side or both sides
- swap the sides.

Aim to do the following, in order.

1. Add or subtract terms on both sides to obtain an equation of the form

   $$\text{a number} \times \text{the unknown} = \text{a number}.$$

2. Divide both sides by the coefficient of the unknown.

As you get more used to solving equations, you'll probably find that you can do the same thing to both sides and simplify the resulting equation all in one step. The worked examples in this unit will usually do this from now on, and you should too, as soon as you feel comfortable with it. Don't try to do too much in one step, however – that can lead to mistakes, and it can also make it hard for other people to follow your working.

Here's an example illustrating this slightly shorter form of working.

Example 23 *Solving an equation efficiently*

Solve the equation

$$3x - 4 = 2 - x.$$

Solution

The equation is:	$3x - 4 = 2 - x$
Add x:	$4x - 4 = 2$
Add 4:	$4x = 6$
Divide by 4:	$x = \dfrac{6}{4} = \dfrac{3}{2}$

The solution is $x = \frac{3}{2}$.

(Check: if $x = \frac{3}{2}$, then

$$\text{LHS} = 3 \times \tfrac{3}{2} - 4 = \tfrac{9}{2} - 4 = \tfrac{9}{2} - \tfrac{8}{2} = \tfrac{1}{2}$$

and

$$\text{RHS} = 2 - \tfrac{3}{2} = \tfrac{4}{2} - \tfrac{3}{2} = \tfrac{1}{2}.$$

Since LHS = RHS, the solution is correct.)

Try this shorter form of working in the following activity.

Activity 41 *Solving equations efficiently*

Solve the following equations.

(a) $4z + 7 = -2z + 6$ (b) $18 = 60 - 7t$

You've seen that, given any correct equation, you can do the same thing to both sides, manipulate the expressions in the equation, or swap the sides of the equation, and you'll obtain another correct equation. If we do any of these things, then we say that we're **rearranging** or *manipulating* the equation. If an equation can be obtained from another equation in this way, then we say that the two equations are **equivalent** or different forms of the same equation.

The ancient Egyptians of around 1800 BC were able to solve linear equations in one unknown. The Rhind papyrus, which dates from about 1650 BC but is a copy of a text from about two centuries earlier, contains a succession of mathematical problems and their solutions, some of which are of this type. However, no general methods of solving linear equations are given, and no letters or other symbols are used to represent unknowns. The Rhind papyrus, which is in the British Museum, is named after the antiquarian Alexander Henry Rhind, who acquired it in Egypt in 1858. It is over five metres long and was first translated in the late nineteenth century.

The ancient Babylonians of around the same time also developed methods for solving linear equations, and for solving problems involving more than one equation. (You'll learn about problems of this type in Unit 7.) Like the ancient Eygptians, the ancient Babylonians didn't use letters or symbols to represent unknowns. The Babylonians lived in Mesopotamia, a region that is now largely Iraq. Our knowledge of Babylonian mathematics is derived from a large number of clay tablets unearthed since the 1850s.

Figure 6 Part of the Rhind papyrus

Figure 7 A Babylonian clay tablet

5.3 Linear equations with fractions and brackets

Some linear equations contain fractions or brackets. You can solve an equation like this by using the usual method of a sequence of steps, in each of which you do the same thing to both sides, simplify one side or both sides or swap the sides. In the first few steps you remove the fractions and brackets, and then you continue as before.

Removing fractions isn't essential (if they're numerical rather than algebraic fractions), but it makes equations easier to manipulate.

Removing a fraction from an equation is often called **clearing** the fraction, and it can be done by multiplying both sides by a suitable number. But you must multiply the *whole* of each side. To see why, consider the following equation, which involves only numbers:

$$\tfrac{1}{2} + 3 = \tfrac{7}{2}. \qquad (12)$$

This equation is correct, as each side has value $\frac{7}{2}$.

If you multiply the whole of each side of the equation by 2, say, then you obtain another correct equation:

$$2\left(\tfrac{1}{2} + 3\right) = 2 \times \tfrac{7}{2}.$$

Each side of this new equation has value 7.

But if you multiply just part of one side by 2, then things go wrong. For example, if you start with equation (12) and multiply the right-hand side by 2, but multiply only the fraction on the left-hand side by 2, then you obtain

$$2 \times \tfrac{1}{2} + 3 = 2 \times \tfrac{7}{2}.$$

This equation is incorrect, because the left- and right-hand sides have values 4 and 7, respectively.

So remember that when you do the same thing to both sides of an equation, you must do it to the whole of each side.

The next example illustrates how to solve an equation that contains fractions and brackets. It's usually best to consider the fractions first, because when you clear a fraction you often need to introduce extra brackets as part of the working. But there are are no hard-and-fast rules. As you become more familiar with solving equations, you'll begin to see the best way to proceed for any particular equation.

Tutorial clip

Example 24 *Solving an equation with fractions and brackets*

Solve the equation

$$4(x-5) = \frac{x}{2} + 8.$$

Solution

The equation is: $4(x-5) = \dfrac{x}{2} + 8$

There's a fraction with denominator 2, so multiply both sides by 2 to clear it. The *whole* of each side must be multiplied by 2, so introduce brackets on the right-hand side.

Multiply by 2: $2 \times 4(x-5) = 2\left(\dfrac{x}{2} + 8\right)$

Simplify: $8(x-5) = 2\left(\dfrac{x}{2} + 8\right)$

Next multiply out the brackets. Multiply *every term* in brackets by the appropriate multiplier.

Multiply out the brackets: $8x - 40 = x + 16$

Now there are no fractions or brackets, so continue in the usual way.

Subtract x: $7x - 40 = 16$

Add 40: $7x = 56$

Divide by 7: $x = 8$

The solution is $x = 8$.

(Check: if $x = 8$, then

$$\text{LHS} = 4(8-5) = 4 \times 3 = 12$$

and

$$\text{RHS} = \frac{8}{2} + 8 = 4 + 8 = 12.$$

Since LHS = RHS, the solution is correct.)

The strategy used in Example 24 is summarised in the box on the next page. It's just the strategy from the previous subsection, with the extra stage of removing fractions and brackets.

Strategy *To solve a linear equation in one unknown*

Carry out a sequence of steps. In each step, do one of the following:

- do the same thing to both sides
- simplify one side or both sides
- swap the sides.

Aim to do the following, in order.

1. Clear any fractions and multiply out any brackets. To clear fractions, multiply both sides by a suitable number.
2. Add or subtract terms on both sides to obtain an equation of the form

$$\text{a number} \times \text{the unknown} = \text{a number}.$$

3. Divide both sides by the coefficient of the unknown.

When you multiply both sides by a number, you must multiply the *whole* of each side.

Activity 42 *Solving equations with fractions and brackets*

Solve the following equations.

(a) $x + 8 = 3(x - 2)$ (b) $\dfrac{2 - x}{7} = 3$ (c) $3(b - 5) = \dfrac{b}{3} + 17$

(d) $3\left(1 + \dfrac{y}{2}\right) = 2(y - 1)$ (e) $\dfrac{1 + a}{2} = 1 + \dfrac{3a}{5}$

Hint for part (e): In this part there are *two* fractions to be cleared. First multiply both sides by a number that will clear one of the fractions, then multiply out any brackets that you have introduced. Then do the same for the other fraction. (A shortcut is to multiply both sides by a number that will clear both fractions at once – the number has to be a common multiple of the two denominators.)

5.4 Using linear equations

Earlier in the unit, you met the idea that some mathematical questions can be answered by using equations. Here's the strategy that can be used.

Strategy *To find an unknown number*

- Represent the number that you want to find by a letter.
- Express the information that you know about the number as an equation.
- Solve the equation.

In this subsection you'll see many different mathematical questions that can be answered by using this strategy. Here's the first example.

Example 25 *Finding an unknown number*

The price of a packet of cereal on special offer has been reduced by 20% and is now £1.80. What's the normal price?

Solution

Represent the number that you want to find by a letter.

Let the normal price, in £, be c.

Express what you know about the number as an equation.

The price has been reduced by 20%, so the reduced price is 80% of the normal price. That is, the reduced price, in £, is

$$\frac{80}{100} \times c, \quad \text{which simplifies to} \quad 0.8c.$$

We know that the reduced price is £1.80, so we obtain the equation

$$0.8c = 1.8.$$

Solve the equation.

We now solve this equation.

Divide by 0.8: $\quad c = \dfrac{1.8}{0.8} = 2.25$

State a conclusion in the context of the question.

So the normal price is £2.25.

(Check: 80% of £2.25 = 0.8 × £2.25 = £1.80.)

When you use the strategy given at the beginning of this subsection, it can be helpful to choose a letter that reminds you of what the letter represents. For example, in Example 25 the letter c stands for cost. But avoid letters that resemble numbers, such as o and l, which look like 0 and 1, respectively. And if the question involves money, then it's best to avoid the letter p, as it can be confused with 'pence'.

Activity 43 *Finding unknown numbers*

Use equations to answer the following questions.

(a) Laura's age, multiplied by four, is 92. How old is Laura?

(b) In a village raffle, 8% of the people who bought a ticket won a prize. There were 16 prizes. How many people bought a ticket?

(c) Rahul celebrated his thirty-fourth birthday in 2008. In what year was he born?

Hint: The equation needed here involves addition rather than multiplication.

(d) Jakub's house is valued at £175 000. House prices in his area have increased by 25% over the last five years. If Jakub's house is typical, what was it worth five years ago?

In the next example there's more than one unknown number. Since the unknown numbers are related, the question can still be answered by using an equation.

Example 26 *Finding related unknown numbers*

Callum, Ewan and Finlay have helped out in their grandfather John's garden, and John has given them £90 to share between them. The money is to be shared according to how much work each grandson has done: Callum and Ewan are to receive three times as much and twice as much, respectively, as Finlay. How much money should each grandson receive?

Solution

There are three numbers (Callum's, Ewan's and Finlay's money) to be found, but once one number has been found, it'll be straightforward to find the other two. So represent just one of the three numbers by a letter.

Let the amount of money that Finlay receives, in £, be m.

Express what you know about the number as an equation.

Then the amounts that Callum and Ewan receive, in £, are $3m$ and $2m$, respectively. The total amount, in £, is

$$m + 2m + 3m.$$

We know that the total amount of money is £90, so we obtain the equation

$$m + 2m + 3m = 90.$$

Solve the equation.

We now solve this equation.

Simplify: $\quad 6m = 90$

Divide by 6: $\quad m = 15$

State a conclusion in the context of the question.

So Finlay receives £15. Hence Callum receives

$$3 \times £15 = £45,$$

and Ewan receives

$$2 \times £15 = £30.$$

(Check: $£15 + £30 + £45 = £90$.)

Activity 44 *Finding related unknown numbers*

Use an equation to answer the following question.

> Lydia and Meena share a flat. Meena has a larger bedroom than Lydia, so they have agreed that the rent that Meena pays should be 1.25 times the rent that Lydia pays. The monthly rent for the flat is £945. How much should each flatmate pay?

Usually the most difficult part of the strategy on page 45 is finding the equation that you need. It's often helpful to write down a rough 'word equation', and then find algebraic expressions to replace the words. This is illustrated in the next example, which is about a 'find-the-age' number puzzle. Puzzles like this aren't so puzzling if you know how to solve equations!

Example 27 *Solving a find-the-age puzzle*

In ten years' time, Matthew will be four times the age he was eight years ago. How old is Matthew?

Solution

Represent the number that you want to find by a letter.

Let Matthew's age be a.

Write down a word equation. To do this, use the information given in the puzzle to find two things that are equal to each other. It's sometimes helpful to look out for the word 'is' (or 'was' or 'will be') in a question or puzzle: this can be the verbal equivalent of an equals sign.

We're told that

$$\text{Matthew's age 10 years from now} = 4 \times \text{Matthew's age 8 years ago.}$$

Now replace the words in the word equation by expressions involving the unknown a. A table can help you to find the right expressions.

Time	Matthew's age
Now	a
10 years from now	$a + 10$
8 years ago	$a - 8$

Replacing the words in the word equation by the expressions from the table gives the equation

$$a + 10 = 4(a - 8).$$

The brackets are essential: Matthew's age eight years ago is $a - 8$, so four times his age eight years ago is $4(a - 8)$, not $4a - 8$.

Solve the equation.

We now solve this equation.

Multiply out the brackets:	$a + 10 = 4a - 32$
Subtract a:	$10 = 3a - 32$
Add 32:	$42 = 3a$
Divide by 3:	$14 = a$

The solution is $a = 14$.

State a conclusion in the context of the question.

That is, Matthew's age is 14.

(Check: In ten years' time Matthew will be 24, and eight years ago he was 6. So his age in ten years' time will be four times the age he was eight years ago.)

Here are some find-the-age puzzles for you to solve. In each part of Activity 45 you might find it easier to compile the age table before writing down the word equation. Do whichever is easier for you.

Activity 45 *Solving find-the-age puzzles*

Solve the following puzzles.

(a) Mariko is four times the age she was 63 years ago. How old is Mariko?

(b) In four years' time, Gregor will be three times the age he was six years ago. How old is Gregor?

(c) Five years ago, Aisha was three times as old as her son Jamil was then. Aisha is 47. How old is Jamil?

Hint: For part (c) you'll need a table with two 'age' columns, one for Aisha's age and one for Jamil's age.

Finally in this unit, we'll use an equation to answer the seemingly complicated question about a charitable donation that you saw in Subsection 1.3. Here's the question.

> Catherine wants to contribute to a charitable cause, using her credit card and a donations website. The donations company states that from each donation, first it will deduct a 2% charge for credit card use, then it will deduct a charge of £3 for use of its website, and then the remaining money will be increased by 22% due to tax payback. How much money (to the nearest penny) must Catherine pay if she wants the cause to receive £40?

To answer this question, let's begin by denoting the amount in £ that Catherine must pay by m. Then we have the following word equation:

amount in £ paid to cause if £m is paid to company = 40.

The next step is to find an expression involving m to replace the left-hand side of this word equation. This can be done by starting with m and using successive steps – it's just like following through a think-of-a-number trick. In the next activity you are asked to do this, and then to solve the resulting equation to find the answer to the question.

Activity 46 *Finding another unknown number*

(a) Follow the steps below to find an expression for the amount in £ paid to the charitable cause if £m is paid to the donations company.

Money paid to company (in £):	m
Multiply by 0.98 (because of the 2% reduction):	
Subtract 3 (the charge for use of the website):	
Multiply by 1.22 (because of tax payback):	

(b) Hence write down an equation whose solution gives the answer to the question above.

(c) Solve the equation found in part (b) and hence state the answer to the question.

Remember that:

- if you deduct 2% of a quantity, then you end up with 0.98 times what you started with
- if you increase a quantity by 22%, then you end up with 1.22 times what you started with.

Percentage increases and decreases were covered in Unit 1.

Now that you've reached the end of this unit, you should have acquired fundamental algebraic skills that will be needed later in this module, and in any further mathematics modules that you study.

Learning checklist

After studying this unit, you should be able to:

- appreciate some of the uses of algebra
- recognise some technical terms used in algebra
- collect like terms
- simplify expressions term by term
- multiply out brackets
- use algebraic fraction notation
- prove some simple facts about numbers
- simplify some formulas
- solve linear equations
- use equations to answer some mathematical questions.

Solutions and comments on Activities

Activity 1

Was your answer *elephant*?

Activity 2

Was your answer *elephant* again? Or perhaps *elk*, or *eel*? See the discussion in the text after the activity.

Activity 3

Substituting $n = 48$ into the formula

$$P = 1.24n - 0.69n$$

gives

$$\begin{aligned} P &= 1.24 \times 48 - 0.69 \times 48 \\ &= 59.52 - 33.12 \\ &= 26.4. \end{aligned}$$

So the profit is £26.40.

Activity 4

Substituting $n = 48$ into the formula

$$P = 0.55n$$

gives

$$P = 0.55 \times 48 = 26.4.$$

So the profit is £26.40.

Activity 5

(a) $\frac{2}{5}T = 24$

(b) Two-fifths of T is 24,
so one-fifth of T is $24 \div 2 = 12$,
so T is $5 \times 12 = 60$.

So there are 60 toddlers in the village.

(You can confirm that this is the right answer by checking that the equation in part (a) is correct when $T = 60$ is substituted in.)

Activity 6

Substitute $a = -2$ and $b = 5$ in each case.

(a)
$$\begin{aligned} \tfrac{5}{2} + a &= \tfrac{5}{2} + (-2) \\ &= \tfrac{5}{2} - 2 \\ &= \tfrac{5}{2} - \tfrac{4}{2} \\ &= \tfrac{1}{2} \end{aligned}$$

(b)
$$\begin{aligned} -a + ab &= -(-2) + (-2) \times 5 \\ &= 2 + (-10) \\ &= 2 - 10 \\ &= -8 \end{aligned}$$

(c)
$$\begin{aligned} ab^2 &= (-2) \times 5^2 \\ &= -2 \times 25 \\ &= -50 \end{aligned}$$

(d)
$$\begin{aligned} b + 3(b - a) &= 5 + 3(5 - (-2)) \\ &= 5 + 3 \times 7 \\ &= 5 + 21 \\ &= 26 \end{aligned}$$

Activity 7

(a) This is correct. Adding three copies of a number together is the same as multiplying it by 3.

(b) This is correct, by an index law. (The index laws were covered in Unit 3.)

(c) This is correct. Multiplying a number by 2 and then dividing by 2 results in the number you started with.

(d) This is incorrect. By an index law, $p^2 \times p^3 = p^5$.

(The statement is correct for $p = 0$ and $p = 1$, but these are the *only* values for which it is correct, so the expressions aren't equivalent.)

(e) This is correct. $2z$ is the same as $z + z$, so $z + 2z$ is the same as $z + z + z$, which is the same as $3z$.

(f) This is correct. Since $6 = 3 \times 2$, multiplying a number by 6 and then dividing by 2 results in 3 times the number you started with.

(g) This is correct. Adding the negative of a number is the same as subtracting the number. (For example, $6 + (-3)$ is the same is $6 - 3$. You met this property of numbers in Unit 1.)

(h) This is incorrect. Multiplying the number 3 by n and then dividing by n gives the result 3.

(The statement is correct for $n = 3$, but this is the *only* value for which it's correct, so the expressions aren't equivalent.)

Activity 8

(a) The expression is

$$\underbrace{+x^3}\;\underbrace{-\,x^2}\;\underbrace{+\,x}\;\underbrace{+\,1}\,.$$

Its terms are $+x^3$, $-x^2$, $+x$ and $+1$.

(b) The expression is

$$\underbrace{+2mn}\;\underbrace{-\,3r}\,.$$

Its terms are $+2mn$ and $-3r$.

(c) The expression is

$$\underbrace{-20p^2q^2}\;\underbrace{+\,\tfrac{1}{4}p}\;\underbrace{-\,18}\;\underbrace{-\,\tfrac{1}{3}q}\,.$$

Its terms are $-20p^2q^2$, $+\frac{1}{4}p$, -18 and $-\frac{1}{3}q$.

Activity 9

(a) The expression is

$$\underbrace{-X}\ \underbrace{+\,20Y}\ \underbrace{-\,5Z}\,.$$

Reversing the order of the terms gives

$$-5Z+20Y-X.$$

(b) The expression is

$$\underbrace{+2u}\ \underbrace{-\,3uv}\,.$$

Reversing the order of the terms gives

$$-3uv+2u.$$

(c) The expression is

$$\underbrace{+4i}\ \underbrace{-\,j}\ \underbrace{+\,5}\,.$$

Reversing the order of the terms gives

$$5-j+4i.$$

(d) The expression is

$$\underbrace{+a}\ \underbrace{-\,b}\ \underbrace{+\,c}\ \underbrace{+\,d}\,.$$

Reversing the order of the terms gives

$$d+c-b+a.$$

Activity 10

(a) The third term is $4y^2$, with coefficient 4.

(b) The second term is $-9\sqrt{q}$, with coefficient -9.

(c) The third term is x^2, with coefficient 1.

(d) The first term is $-a^2b$, with coefficient -1.

(e) The term in m^2 is $-3m^2$, with coefficient -3.

(f) The term in b is $2b$, with coefficient 2. (The term b^2 is a term in b^2, not b.)

Activity 11

(a) There is no constant term.

(b) There is a constant term, -7.

(c) There is a constant term, $5\sqrt{2}$.

(d) There is no constant term.

(e) There is a constant term, 1.

(f) There is no constant term.

Activity 12

(a) These are unlike terms: the first is a term in b, and the second is a term in b^2.

(b) These are like terms: both are terms in D.

(c) These are like terms: both are terms in z. (The first term has coefficient 1, and the second has coefficient -1.)

(d) These are unlike terms: the first is a constant term, and the second is a term in m.

Activity 13

(a) $8A+7A=(8+7)A=15A$

(b) $-5d+8d-2d=(-5+8-2)d=1d=d$

($1d$ is usually written as d.)

(c) $-7z+z=-7z+1z=(-7+1)z=-6z$

(d)
$$\begin{aligned}1.4pq+0.7pq-pq&=1.4pq+0.7pq-1pq\\&=(1.4+0.7-1)pq\\&=1.1pq\end{aligned}$$

(e) $\frac{1}{2}n^2-\frac{1}{3}n^2=\frac{3}{6}n^2-\frac{2}{6}n^2=\left(\frac{3}{6}-\frac{2}{6}\right)n^2=\frac{1}{6}n^2$

(You should give the exact answer, $\frac{1}{6}n^2$, not an approximation such as $0.167n^2$.)

Activity 14

(a) These are like terms: both are terms in ab.

(b) These are like terms: both are terms in rst.

(c) These are like terms: both are terms in xy. (The second term can be written as $-3xy$.)

(d) These are like terms: both are terms in ac^2. (The first term can be written as $4ac^2$.)

(e) These are like terms: both are terms in abc. (The second term can be written as abc.)

(f) These are unlike terms. If we write the second term with the letters in alphabetical order, then it's $9cd^2$. So the first term is a term in c^2d (that is, $c\times c\times d$), and the second is a term in cd^2 (that is, $c\times d\times d$).

(g) These are unlike terms: the first is a term in A^2, and the second is a term in a^2.

(h) These are unlike terms: the first is a term in fh, and the second is a term in gh.

(i) These are like terms, as they're both constant terms.

Activity 15

(a)
$$\begin{aligned}&4A-3B+3C+5A+2B-A\\&=4A+5A-A-3B+2B+3C\\&=8A-B+3C\end{aligned}$$

(b)
$$\begin{aligned}&-8v+7-5w-2v-8\\&=-8v-2v-5w+7-8\\&=-10v-5w-1\end{aligned}$$

(c)
$$\begin{aligned}&20y^2+10xy-10y^2-5y-5xy\\&=20y^2-10y^2+10xy-5xy-5y\\&=10y^2+5xy-5y\end{aligned}$$

(d)
$$\begin{aligned}&-4ef+8e^2f+10fe-3f^2e\\&=-4ef+8e^2f+10ef-3ef^2\\&=-4ef+10ef+8e^2f-3ef^2\\&=6ef+8e^2f-3ef^2\end{aligned}$$

(e) $\frac{1}{2}a + \frac{1}{3}b + 2a + \frac{1}{4}b$

$= \frac{1}{2}a + 2a + \frac{1}{3}b + \frac{1}{4}b$

$= \frac{1}{2}a + \frac{4}{2}a + \frac{4}{12}b + \frac{3}{12}b$

$= \frac{5}{2}a + \frac{7}{12}b$

Activity 16

(a) $2a^3 - 3a - 2a^3 - 3a = -6a$

(b) $2m + n - 5m + 2n + 3m = 3n$

(c) $b + 2b + 3b - 6b = 0$

Activity 17

(a) The formula is

$A = 10c + 2a.$

(b) The formula is

$T = 7c + 14a.$

(c) The formula is

$C = 10c + 2a + 7c + 14a.$

(d) Collecting like terms gives

$C = 17c + 16a.$

(e) Substituting $c = 22$ and $a = 10$ in the formula found in part (d) gives

$C = 17 \times 22 + 16 \times 10 = 374 + 160 = 534.$

The cost of the trip is £534.

Activity 18

(a) $y \times z \times 6 \times x \times 4 = 24xyz$

(b) $7p \times 2qr = 14pqr$

(c) $QR \times G \times 5F = 5FGQR$

(d) $2 \times a \times a \times 3 \times a = 6a^3$

(e) $m \times n \times m \times 4 = 4m^2n$

(f) $5y \times 2yx = 10xy^2$

(g) $4AB \times 4AB = 16A^2B^2$

Activity 19

(a) $8P^8 \times 5P = 40P^9$

(b) $2c^{10}d^3 \times 2c^2d^3 = 4c^{12}d^6$

Activity 20

(a) $9X \times (-XY) = -9X^2Y$

(b) $3s \times \frac{1}{3}r = 1rs = rs$

(c) $-3a^3 \times (-4a^4) = +12a^7 = 12a^7$

(d) $-2pq \times (-3qp^2) = +6p^3q^2 = 6p^3q^2$

(e) $-0.5g \times 2f^5 = -1f^5g = -f^5g$

(f) $-a \times b \times (-c) \times (-d) = -abcd$

(g) $(-x) \times (-y) \times (-x^2) \times (-4y)$

$= +4x^3y^2 = 4x^3y^2$

(h) $(-3cd)^2 = (-3cd) \times (-3cd) = +9c^2d^2 = 9c^2d^2$

(i) $-(3cd)^2 = -(3cd \times 3cd) = -9c^2d^2$

You could do parts (c), (d), (g) and (h) in one step if you prefer.

Activity 21

(a) $+(-ab) = -ab$

(b) $-(-6x^2) = +6x^2 = 6x^2$

(c) $-(2M^4) = -2M^4$

(d) $+(-7y) = -7y$

(e) $+(5p) = +5p = 5p$

(f) $-\left(-\frac{3}{4}n\right) = \frac{3}{4}n$

Activity 22

(a) $\underbrace{5m \times 2m}\ \underbrace{- 2n \times n^2} = 10m^2 - 2n^3$

(b) $\underbrace{3p \times 2q}\ \underbrace{+ 2r \times (-7p)} = 6pq - 14pr$

(c) $\underbrace{2P}\ \underbrace{- (-3Q)}\ \underbrace{+ (-P)}\ \underbrace{+ (2Q)}$

$= 2P + 3Q - P + 2Q$

$= P + 5Q$

(d) $\underbrace{3 \times (-2a)}\ \underbrace{- 1c^2}\ \underbrace{+ 9ac} = -6a - c^2 + 9ac$

(Only the first and second terms were simplified. The third term was already in its simplest form.)

(e) $\underbrace{4s \times \frac{1}{2}rst}\ \underbrace{- 2(-\frac{1}{2}s)} = 2rs^2t + s$

(f) $\underbrace{-5xy}\ \underbrace{+ (-3y \times x^2)}\ \underbrace{- (-y^2)}$

$= -5xy - 3x^2y + y^2$

(Only the second and third terms were simplified. The first term was already in its simplest form.)

(g) $\underbrace{-3r \times (-2r)}\ \underbrace{- (-2r \times r)}\ \underbrace{+ (r^2 \times 9)}$

$= 6r^2 + 2r^2 + 9r^2$

$= 17r^2$

Activity 23

(a) $3p(pq + 4) = 3p \times pq + 3p \times 4 = 3p^2q + 12p$

(b) $x^2(x + 2y) = x^2 \times x + x^2 \times 2y = x^3 + 2x^2y$

Activity 24

(a) $f(e+5g) = ef + 5fg$

(b) $5(2A+B) = 10A + 5B$

(c) $3c(4c+2d) = 12c^2 + 6cd$

(d) $(a-b)c^2 = ac^2 - bc^2$

(e) $2y(x+2y+4z) = 2xy + 4y^2 + 8yz$

(f) $2\left(\frac{1}{2}A^2 + \frac{3}{2}\right) = A^2 + 3$

(g) $a(x+y)z = axz + ayz$

(h) $2b(b^2 + 2b^4) = 2b^3 + 4b^5$

Activity 25

(a) $p(q-r) = pq - pr$

(b) $7a(-4a+3b) = -28a^2 + 21ab$

(c) $6(0.2a - 0.3b + 1.4) = 1.2a - 1.8b + 8.4$

(d) $10(\frac{1}{2}n + \frac{1}{5}) = 5n + 2$

(e) $-3(x-2y) = -3x + 6y$

(f) $-b^2(-a+b) = ab^2 - b^3$

Activity 26

(a) $\underbrace{f}\ \underbrace{+g(f+h)} = f + fg + gh$

(b) $\underbrace{x}\ \underbrace{-y(x+2y)} = x - xy - 2y^2$

(c) $\underbrace{2p}\ \underbrace{-3q(-3p+q)} = 2p + 9pq - 3q^2$

(d) $\underbrace{-2(a+b)}\ \underbrace{+4(a-b)} = -2a - 2b + 4a - 4b$
$= 2a - 6b$

(e) $\underbrace{2aE}\ \underbrace{-3E(-E-5a)} = 2aE + 3E^2 + 15aE$
$= 17aE + 3E^2$

(f) $\underbrace{(d-c)c}\ \underbrace{-c^2} = dc - c^2 - c^2$
$= cd - 2c^2$

Activity 27

(a) $-(4f - g^3) = -4f + g^3$

(b) $-(-x + 7y - 8z + 6) = x - 7y + 8z - 6$

(c) $\underbrace{2(a-b)}\ \underbrace{+(c-2d)} = 2a - 2b + c - 2d$

(d) $\underbrace{r}\ \underbrace{+(-2s-r)} = r - 2s - r = -2s$

(e) $\underbrace{-A}\ \underbrace{+B}\ \underbrace{-(-3A+4B)}$
$= -A + B + 3A - 4B = 2A - 3B$

(f) $\underbrace{-(-t-w)}\ \underbrace{+(-t+w)} = t + w - t + w$
$= 2w$

(g) $\underbrace{-(L+2M)}\ \underbrace{-(-M)} = -L - 2M + M$
$= -L - M$

Activity 28

(a) $(a+b) \div 3 = \dfrac{a+b}{3}$

(b) $a + b \div 3 = a + \dfrac{b}{3}$

(c) $(x+2) \div (y+3) = \dfrac{x+2}{y+3}$

(d) $(x+2) \div y + 3 = \dfrac{x+2}{y} + 3$

(e) $x + 2 \div (y+3) = x + \dfrac{2}{y+3}$

(f) $x + 2 \div y + 3 = x + \dfrac{2}{y} + 3$

Activity 29

(a) $6\left(1 + \dfrac{h}{2}\right) = 6 + \dfrac{6h}{2} = 6 + 3h$

(b) $6\left(\dfrac{1+h}{2}\right) = \dfrac{6}{2}(1+h) = 3(1+h) = 3 + 3h$

Activity 30

(a) $\dfrac{A-6B}{3} = \dfrac{A}{3} - \dfrac{6B}{3} = \dfrac{A}{3} - 2B$

(b) $\dfrac{10z^2 + 5z - 20}{5} = \dfrac{10z^2}{5} + \dfrac{5z}{5} - \dfrac{20}{5}$
$= 2z^2 + z - 4$

(c) $\dfrac{3A^2 + A}{A} = \dfrac{3A^2}{A} + \dfrac{A}{A} = 3A + 1$

Activity 31

(a) For example, here's the trick with starting number 7.

Think of a number:	7
Multiply it by 3:	21
Add 2:	23
Double the result:	46
Add 2:	48
Divide by 6:	8
Take away the number you first thought of:	1

The answer is 1.

(b) Here's the trick starting with n.

Think of a number:	n
Multiply it by 3:	$3n$
Add 2:	$3n + 2$
Double the result:	$2(3n+2) = 6n + 4$
Add 2:	$6n + 4 + 2 = 6n + 6$
Divide by 6:	$\dfrac{6n+6}{6} = \dfrac{6n}{6} + \dfrac{6}{6} = n + 1$

Take away the number
you first thought of: $n + 1 - n = 1$

So the trick always gives the answer 1.

Activity 32

(a) The cost for the first 200 leaflets is £175. The remaining number of leaflets is $n - 200$, and the cost in £ for these leaflets is

$$(n - 200) \times 0.25 = 0.25(n - 200).$$

The total cost in £ is

$$175 + 0.25(n - 200).$$

So the formula is

$$C = 175 + 0.25(n - 200).$$

(b) The formula can be simplified as follows:

$$\begin{aligned} C &= 175 + 0.25(n - 200) \\ &= 175 + 0.25n - 50 \\ &= 125 + 0.25n. \end{aligned}$$

The formula is

$$C = 125 + 0.25n.$$

(c) If $n = 450$, then

$$\begin{aligned} C &= 125 + 0.25 \times 450 \\ &= 125 + 112.5 \\ &= 237.5. \end{aligned}$$

So the cost of printing 450 leaflets is £237.50.

Activity 33

For example, the consecutive integers 1, 2 and 3 add up to 6, which is divisible by 3. Then, for example, the consecutive integers 7, 8 and 9 add up to 24, which is also divisible by 3.

Activity 34

(a) For example, $5 + 7 + 9 = 21$, which is divisible by 3.

(b) For example, $20 + 22 + 24 = 66$, which is divisible by 3.

(c) Represent the first of the three integers by n. Then the other two integers are $n + 2$ and $n + 4$. So their sum is

$$\begin{aligned} n + (n + 2) + (n + 4) &= n + n + 2 + n + 4 \\ &= 3n + 6. \end{aligned}$$

Dividing by 3 gives

$$\frac{3n + 6}{3} = \frac{3n}{3} + \frac{6}{3} = n + 2.$$

Now $n + 2$ is an integer, because n is an integer. So dividing the sum of the three numbers by 3 gives an integer. That is, the sum is divisible by 3.

Activity 35

Represent the first of the four integers by n. Then the other three integers are $n + 1$, $n + 2$ and $n + 3$. So their sum is

$$\begin{aligned} n &+ (n + 1) + (n + 2) + (n + 3) \\ &= n + n + 1 + n + 2 + n + 3 \\ &= 4n + 6. \end{aligned}$$

Dividing by 4 gives

$$\frac{4n + 6}{4} = \frac{4n}{4} + \frac{6}{4} = n + \tfrac{3}{2}.$$

Since n is an integer, $n + \frac{3}{2}$ is *not* an integer. So the sum is not divisible by 4.

Activity 36

(a) For example, if you choose the first example in the margin, then the first integer is 15 and the amount by which the second and third integers are more than the one before is 4. The three integers are 15, 19 and 23, and their sum is $15 + 19 + 23 = 57$, which is divisible by 3, since $57 \div 3 = 19$.

(b) An expression for the second integer is $n + d$, and an expression for the third integer is $n + d + d$, which simplifies to $n + 2d$.

(c) An expression for the sum of the three integers is

$$\begin{aligned} n + (n + d) + (n + 2d) &= n + n + d + n + 2d \\ &= 3n + 3d. \end{aligned}$$

Dividing the expression in part (c) by by 3 gives

$$\frac{3n + 3d}{3} = \frac{3n}{3} + \frac{3d}{3} = n + d.$$

Since n and d are both integers, so is $n + d$. So dividing the sum of the three integers by 3 gives an integer. That is, the sum is divisible by 3.

Activity 37

(a) If $p = 3$, then

$$\text{LHS} = 4 \times 3 = 12 = \text{RHS},$$

so $p = 3$ is a solution. Hence the statement is true.

(b) If $A = 2$, then

$$\text{LHS} = 10 - 2 \times 2 = 10 - 4 = 6$$

and

$$\text{RHS} = 1 + 2 = 3.$$

Thus the LHS and the RHS are not equal, so $A = 2$ is not a solution. Hence the statement is false.

(c) If $z = -5$, then

$$\text{LHS} = 4 \times (-5) + 2 = -20 + 2 = -18$$

and

$$\text{RHS} = 3(-5 - 1) = 3 \times (-6) = -18.$$

Thus LHS = RHS, so $z = -5$ is a solution. Hence the statement is true.

Activity 38

(a) The equation is: $5x = 20$

Divide by 5: $\dfrac{5x}{5} = \dfrac{20}{5}$

Simplify: $x = 4$

The solution is $x = 4$.

(Check: if $x = 4$, then

$\text{LHS} = 5 \times 4 = 20 = \text{RHS}$,

so the solution is correct.)

(b) The equation is: $t - 6 = 7$

Add 6: $t - 6 + 6 = 7 + 6$

Simplify: $t = 13$

The solution is $t = 13$.

(Check: if $t = 13$, then

$\text{LHS} = 13 - 6 = 7 = \text{RHS}$,

so the solution is correct.)

(c) The equation is: $x + 4 = 1$

Subtract 4: $x + 4 - 4 = 1 - 4$

Simplify: $x = -3$

The solution is $x = -3$.

(Check: if $x = -3$, then

$\text{LHS} = -3 + 4 = 1 = \text{RHS}$,

so the solution is correct.)

(d) The equation is: $\dfrac{z}{2} = 8$

Multiply by 2: $\dfrac{z}{2} \times 2 = 8 \times 2$

Simplify: $z = 16$

The solution is $z = 16$.

(Check: if $z = 16$, then

$\text{LHS} = \dfrac{16}{2} = 8 = \text{RHS}$,

so the solution is correct.)

(e) The equation is: $x - 1.7 = 3$

Add 1.7: $x - 1.7 + 1.7 = 3 + 1.7$

Simplify: $x = 4.7$

The solution is $x = 4.7$.

(Check: if $x = 4.7$, then

$\text{LHS} = 4.7 - 1.7 = 3 = \text{RHS}$,

so the solution is correct.)

(f) The equation is: $3X = 4$

Divide by 3: $\dfrac{3X}{3} = \dfrac{4}{3}$

Simplify: $X = \frac{4}{3}$

The solution is $X = \frac{4}{3}$.

(Give the exact answer, $\frac{4}{3}$, not an approximation such as 1.33. You could convert the top-heavy fraction $\frac{4}{3}$ to the mixed number $1\frac{1}{3}$, but you don't have to do that.)

(Check: if $X = \frac{4}{3}$, then

$\text{LHS} = 3 \times \frac{4}{3} = 4 = \text{RHS}$,

so the solution is correct.)

(g) The equation is: $-2y = 10$

Divide by -2: $\dfrac{-2y}{-2} = \dfrac{10}{-2}$

Simplify: $y = -5$

The solution is $y = -5$.

(Check: if $y = -5$, then

$\text{LHS} = -2 \times (-5) = 10 = \text{RHS}$,

so the solution is correct.)

(h) The equation is: $\dfrac{c}{-5} = -6$

Multiply by -5: $\dfrac{c}{-5} \times (-5) = (-6) \times (-5)$

Simplify: $c = 30$

The solution is $c = 30$.

(Check: if $c = 30$, then

$\text{LHS} = \dfrac{30}{-5} = -6 = \text{RHS}$,

so the solution is correct.)

(i) The equation is: $-m = 12$

Multiply by -1: $-m \times (-1) = 12 \times (-1)$

Simplify: $m = -12$

The solution is $m = -12$.

(Check: if $m = -12$, then

$\text{LHS} = -(-12) = 12 = \text{RHS}$,

so the solution is correct.)

Activity 39

(a) The equation is: $9x = 12 + 5x$

Subtract $5x$: $9x - 5x = 12 + 5x - 5x$

Simplify: $4x = 12$

Divide by 4: $\dfrac{4x}{4} = \dfrac{12}{4}$

Simplify: $x = 3$

The solution is $x = 3$.

(Check: if $x = 3$, then

$\text{LHS} = 9 \times 3 = 27$

and

$\text{RHS} = 12 + 5 \times 3 = 12 + 15 = 27$.

Since LHS = RHS, the solution is correct.)

(b) The equation is: $6x + 8 = 2$
Subtract 8: $6x + 8 - 8 = 2 - 8$
Simplify: $6x = -6$
Divide by 6: $\dfrac{6x}{6} = \dfrac{-6}{6}$
Simplify: $x = -1$

The solution is $x = -1$.

(Check: if $x = -1$, then
$\text{LHS} = 6 \times (-1) + 8 = -6 + 8 = 2 = \text{RHS}$,
so the solution is correct.)

(c) The equation is: $9x = 6 - 3x$
Add $3x$: $9x + 3x = 6 - 3x + 3x$
Simplify: $12x = 6$
Divide by 12: $\dfrac{12x}{12} = \dfrac{6}{12}$
Simplify: $x = \frac{1}{2}$

The solution is $x = \frac{1}{2}$.

(Check: if $x = \frac{1}{2}$, then
$\text{LHS} = 9 \times \frac{1}{2} = \frac{9}{2}$
and
$\text{RHS} = 6 - 3 \times \frac{1}{2} = \frac{12}{2} - \frac{3}{2} = \frac{9}{2}$.
Since LHS = RHS, the solution is correct.)

Activity 40

(a) The equation is: $3x + 2 = x + 10$
Subtract x: $3x + 2 - x = x + 10 - x$
Simplify: $2x + 2 = 10$
Subtract 2: $2x + 2 - 2 = 10 - 2$
Simplify: $2x = 8$
Divide by 2: $\dfrac{2x}{2} = \dfrac{8}{2}$
Simplify: $x = 4$

The solution is $x = 4$.

(Check: if $x = 4$, then
$\text{LHS} = 3 \times 4 + 2 = 12 + 2 = 14$
and
$\text{RHS} = 4 + 10 = 14$.
Since LHS = RHS, the solution is correct.)

(b) The equation is: $5x + 9 = -x - 3$
Add x: $5x + 9 + x = -x - 3 + x$
Simplify: $6x + 9 = -3$
Subtract 9: $6x + 9 - 9 = -3 - 9$
Simplify: $6x = -12$
Divide by 6: $\dfrac{6x}{6} = \dfrac{-12}{6}$
Simplify: $x = -2$

The solution is $x = -2$.

(Check: if $x = -2$, then
$\text{LHS} = 5 \times (-2) + 9 = -10 + 9 = -1$
and
$\text{RHS} = -(-2) - 3 = 2 - 3 = -1$.
Since LHS = RHS, the solution is correct.)

Activity 41

(a) The equation is: $4z + 7 = -2z + 6$
Add $2z$: $6z + 7 = 6$
Subtract 7: $6z = -1$
Divide by 6: $z = \dfrac{-1}{6} = -\frac{1}{6}$

The solution is $z = -\frac{1}{6}$.

(Check: if $z = -\frac{1}{6}$, then
$\text{LHS} = 4(-\frac{1}{6}) + 7$
$= -\frac{2}{3} + 7 = -\frac{2}{3} + \frac{21}{3} = \frac{19}{3}$
and
$\text{RHS} = -2(-\frac{1}{6}) + 6$
$= \frac{1}{3} + 6 = \frac{1}{3} + \frac{18}{3} = \frac{19}{3}$.
Since LHS = RHS, the solution is correct.)

(b) The equation is: $18 = 60 - 7t$
Swap the sides: $60 - 7t = 18$
Subtract 60: $-7t = -42$
Divide by -7: $t = 6$

The solution is $t = 6$.

(An alternative way to solve the equation is as follows.
The equation is: $18 = 60 - 7t$
Add $7t$: $18 + 7t = 60$
Subtract 18: $7t = 42$
Divide by 7: $t = 6$)

(Check: if $t = 6$, then
$\text{RHS} = 60 - 7 \times 6 = 60 - 42 = 18 = \text{LHS}$,
so the solution is correct.)

Activity 42

(a) The equation is: $x + 8 = 3(x - 2)$
Multiply out the brackets: $x + 8 = 3x - 6$
Subtract x: $8 = 2x - 6$
Add 6: $14 = 2x$
Divide by 2: $7 = x$

The solution is $x = 7$.

(Check: if $x = 7$, then $\text{LHS} = 7 + 8 = 15$ and $\text{RHS} = 3(7 - 2) = 3 \times 5 = 15$, so the solution is correct.)

(b) The equation is: $\dfrac{2-x}{7} = 3$

Multiply by 7: $2 - x = 21$

Subtract 2: $-x = 19$

Multiply by -1: $x = -19$

The solution is $x = -19$.

(Check: if $x = -19$, then

$$\text{LHS} = \frac{2-(-19)}{7} = \frac{21}{7} = 3 = \text{RHS},$$

so the solution is correct.)

(c) The equation is: $3(b-5) = \dfrac{b}{3} + 17$

Multiply by 3: $3 \times 3(b-5) = 3\left(\dfrac{b}{3} + 17\right)$

Simplify: $9(b-5) = 3\left(\dfrac{b}{3} + 17\right)$

Multiply out the brackets: $9b - 45 = b + 51$

Subtract b: $8b - 45 = 51$

Add 45: $8b = 96$

Divide by 8: $b = 12$

The solution is $b = 12$.

(Check: if $b = 12$, then

$$\text{LHS} = 3(12-5) = 3 \times 7 = 21$$

and

$$\text{RHS} = \frac{12}{3} + 17 = 4 + 17 = 21.$$

Since LHS = RHS, the solution is correct.)

(d) The equation is: $3\left(1 + \dfrac{y}{2}\right) = 2(y-1)$

Multiply by 2: $2 \times 3\left(1 + \dfrac{y}{2}\right) = 2 \times 2(y-1)$

Simplify: $6\left(1 + \dfrac{y}{2}\right) = 4(y-1)$

Multiply out the brackets: $6 + 3y = 4y - 4$

Subtract $3y$: $6 = y - 4$

Add 4: $10 = y$

The solution is $y = 10$.

(Check: if $y = 10$, then

$$\text{LHS} = 3\left(1 + \frac{10}{2}\right) = 3(1+5) = 3 \times 6 = 18$$

and

$$\text{RHS} = 2(10-1) = 2 \times 9 = 18.$$

Since LHS = RHS, the solution is correct.)

(e) The equation is: $\dfrac{1+a}{2} = 1 + \dfrac{3a}{5}$

Multiply by 2: $1 + a = 2\left(1 + \dfrac{3a}{5}\right)$

Multiply out the brackets: $1 + a = 2 + \dfrac{6a}{5}$

Multiply by 5: $5(1+a) = 5\left(2 + \dfrac{6a}{5}\right)$

Multiply out the brackets: $5 + 5a = 10 + 6a$

Subtract $5a$: $5 = 10 + a$

Subtract 10: $-5 = a$

The solution is $a = -5$.

(Check: if $a = -5$, then

$$\text{LHS} = \frac{1+(-5)}{2} = \frac{1-5}{2} = \frac{-4}{2} = -2$$

and

$$\text{RHS} = 1 + \frac{3 \times (-5)}{5} = 1 + \frac{-15}{5} = 1 + (-3) = -2.$$

Since LHS = RHS, the solution is correct.)

(If instead you begin by multiplying by a number that will remove both fractions, the working begins as follows.

The equation is: $\dfrac{1+a}{2} = 1 + \dfrac{3a}{5}$

Multiply by 10: $10\left(\dfrac{1+a}{2}\right) = 10\left(1 + \dfrac{3a}{5}\right)$

Simplify the LHS: $5(1+a) = 10\left(1 + \dfrac{3a}{5}\right)$

Multiply out the brackets: $5 + 5a = 10 + 6a$

The working continues as above.)

Activity 43

(a) Let Laura's age be a. Then four times her age is $4a$. We're told that this is equal to 92, so we obtain the equation

$$4a = 92.$$

We now solve this equation.

Divide by 4: $a = \dfrac{92}{4} = 23$

So Laura is 23.

(Check: $4 \times 23 = 92$.)

(b) Let the number of people who bought a ticket be n. We're told that 8% of this number is 16, so we obtain the equation

$$\frac{8}{100} \times n = 16.$$

We now solve this equation.

Multiply by 100: $\quad 8n = 1600$

Divide by 8: $\quad n = 200$

So 200 people bought a ticket.

(Check: $\dfrac{8}{100} \times 200 = 16$.)

(The initial equation could have been simplified to $0.08n = 16$. Then it could have been solved by dividing both sides by 0.08.)

(c) Let the year of Rahul's birth be y. Then the year in which he had his thirty-fourth birthday is $y + 34$. We're told that this is 2008, so we obtain the equation

$$y + 34 = 2008.$$

We now solve this equation.

Subtract 34: $\quad y = 1974$

So Rahul was born in 1974.

(Check: $1974 + 34 = 2008$.)

(d) Let the value, in £, of Jakub's house five years ago be v.

Then its value now, in £, is

$$\frac{125}{100} \times v = 1.25v.$$

We know that this value is £175 000, so we obtain the equation

$$1.25v = 175\,000.$$

We now solve this equation.

Divide by 1.25: $\quad v = \dfrac{175\,000}{1.25} = 140\,000$

So the value five years ago was £140 000.

(Check: $1.25 \times$ £140 000 $=$ £175 000.)

Activity 44

Let Lydia's share of the rent, in £, be r. Then Meena's share, in £, is $1.25r$. The total rent, in £, is

$$r + 1.25r.$$

The total rent is £945, so we obtain the equation

$$r + 1.25r = 945.$$

We now solve this equation.

Simplify: $\quad 2.25r = 945$

Divide by 2.25: $\quad r = \dfrac{945}{2.25} = 420$

So Lydia's share is £420. Hence Meena's share is

$$1.25 \times £420 = £525.$$

(Check: £420 + £525 = £945.)

Activity 45

(a) Let Mariko's age be a.

We're told that

Mariko's age now $= 4 \times$ her age 63 years ago.

Time	Mariko's age
Now	a
63 years ago	$a - 63$

Replacing the words in the word equation by the expressions from the table gives the equation

$$a = 4(a - 63).$$

We now solve this equation.

Multiply out the brackets: $\quad a = 4a - 252$

Subtract a: $\quad 0 = 3a - 252$

Add 252: $\quad 252 = 3a$

Divide by 3: $\quad 84 = a$

So Mariko is 84.

(Check: 63 years ago, Mariko's age was $84 - 63 = 21$. Since $4 \times 21 = 84$, Mariko is four times the age she was 63 years ago.)

(b) Let Gregor's age be a.

We're told that

Gregor's age in 4 years' time
$= 3 \times$ his age 6 years ago.

Time	Gregor's age
Now	a
4 years from now	$a + 4$
6 years ago	$a - 6$

Replacing the words in the word equation by the expressions from the table gives the equation

$$a + 4 = 3(a - 6).$$

We now solve this equation.

Multiply out the brackets: $\quad a + 4 = 3a - 18$

Subtract a: $\quad 4 = 2a - 18$

Add 18: $\quad 22 = 2a$

Divide by 2: $\quad 11 = a$

So Gregor is 11.

(Check: In four years' time, Gregor will be 15, and six years ago he was 5. So his age in four years' time will be three times the age he was six years ago.)

(c) Let Jamil's age be a.

We're told that

Aisha's age 5 years ago
$= 3 \times$ Jamil's age 5 years ago.

Time	Aisha's age	Jamil's age
Now	47	a
5 years ago	42	$a - 5$

Replacing the words in the word equation by the expressions from the table gives the equation

$$42 = 3(a - 5).$$

We now solve this equation.

Multiply out the brackets: $42 = 3a - 15$

Add 15: $57 = 3a$

Divide by 3: $19 = a$

So Jamil is 19.

(Check: Five years ago Aisha was 42, and Jamil was 14. Since $3 \times 14 = 42$, at that time Aisha was three times as old as Jamil.)

Activity 46

(a) Money paid to company (in £): m

Multiply by 0.98 (because of the 2% reduction): $0.98m$

Subtract 3 (the charge for use of the website): $0.98m - 3$

Multiply by 1.22 (because of tax payback): $1.22(0.98m - 3) = 1.1956m - 3.66$

(b) The equation is

$$1.1956m - 3.66 = 40.$$

(c) Start with the equation in part (b).

Add 3.66: $1.1956m = 43.66$

Divide by 1.1956: $m = \dfrac{43.66}{1.1956} = 36.52$ (to 2 d.p.)

So Catherine must donate £36.52.

(Check: $0.98 \times$ £36.52 = £35.7896.
£35.7896 − £3 = £32.7896.
$1.22 \times$ £32.7896 ≈ £40.)